BILINEAR HILBERT TRANSFORMS ALONG CURVES
I. THE MONOMIAL CASE

XIAOCHUN LI

ABSTRACT. We establish an $L^2 \times L^2$ to L^1 estimate for the bilinear Hilbert transform along a curve defined by a monomial. Our proof is closely related to multilinear oscillatory integrals.

1. INTRODUCTION

Let $d \geq 2$ be a positive integer. We consider the bilinear Hilbert transform along a curve $\Gamma(t) = (t, t^d)$ defined by

$$(1.1) \qquad H_\Gamma(f, g)(x) = p.v. \int_{\mathbb{R}} f(x - t) g(x - t^d) \frac{dt}{t} \,,$$

where f, g are Schwartz functions on $\mathbb{R}$.

The main theorem we prove in this paper is

Theorem 1.1. *The bilinear Hilbert transform along the curve $\Gamma(t) = (t, t^d)$ can be extended to a bounded operator from $L^2 \times L^2$ to L^1.*

Remark 1.1. *It can be shown, with a little modification of our method, that the bilinear Hilbert transforms along polynomial curves $(t, P(t))$ are bounded from $L^p \times L^q$ to L^r whenever $(1/p, 1/q, 1/r)$ is in the closed convex hull of $(1/2, 1/2, 1)$, $(1/2, 0, 1/2)$ and $(0, 1/2, 1/2)$.*

This problem is motivated by the Hilbert transform along a curve $\Gamma = (t, \gamma(t))$ defined by

$$H_\Gamma(f)(x_1, x_2) = p.v. \int_{\mathbb{R}} f(x_1 - t, x_2 - \gamma(t)) \frac{dt}{t} \,,$$

and the bilinear Hilbert transform defined by

$$H(f, g)(x) = p.v. \int_{\mathbb{R}} f(x - t) g(x + t) \frac{dt}{t} \,.$$

Among various curves, one simple model case is the parabola (t, t^2) in the two dimensional plane. This work was initiated by Fabes and Riviere [7] in order to study the regularity of parabolic differential equations. In the last thirty years, considerable work on this type of problems had been done. A nice survey on this type of operators was written by Stein and Wainger [27]. For the curves on homogeneous nilpotent Lie

groups, the L^p estimates were established by Christ [2]. The work for the Hilbert transform along more general curves with certain geometric conditions such as the "flat" case can be found in Christ, Duoandikoetxea and J. L. Rubio de Francia, and Nagel, Vance, Wainger and Weinberg's papers [6, 3, 22]. The general results were established recently in [5] for the singular Radon transforms and their maximal analogues over smooth submanifolds of $\mathbb{R}^n$ with some curvature conditions.

In recent years there has been a very active trend of harmonic analysis using time-frequency analysis to deal with multi-linear operators. A breakthrough on the bilinear Hilbert transform was made by Lacey and Thiele [17, 18]. Following Lacey and Thiele's work, the field of multi-linear operators has been actively developed, to the point that some of the most interesting open questions have a strong connection to some kind of non-abelian analysis. For instance, the tri-linear Hilbert transform

$$p.v. \int f_1(x+t)f_2(x+2t)f_3(x+3t)\frac{dt}{t}$$

has a hidden quadratic modulation symmetry which must be accounted for in any proposed method of analysis. This non-abelian character is explicit in the work of B. Kra and B. Host [14] who characterize the characteristic factor of the corresponding ergodic averages

$$N^{-1}\sum_{n=1}^{N} f_1(T^n)f_2(T^{2n})f_3(T^{3n}) \longrightarrow \prod_{j=1}^{3}\mathbb{E}(f_j \mid \mathcal{N})$$

Here, $(X, \mathcal{A}, \mu, T)$ is a measure preserving system, $\mathcal{N} \subset \mathcal{A}$ is the sigma-field which describes the characteristic factor. In this case, it arises from certain 2-step nilpotent groups. The limit above is in the sense of L^2-norm convergence, and holds for all bounded f_1, f_2, f_3.

The ergodic analog of the bilinear Hilbert transform along a parabola is the non-conventional bi-linear average

$$N^{-1}\sum_{n=1}^{N} f_1(T^n)f_2(T^{n^2}) \longrightarrow \prod_{j=1}^{2}\mathbb{E}(f_j \mid \mathcal{K}_{\text{profinite}})$$

where $\mathcal{K}_{\text{profinite}} \subset \mathcal{A}$ is the profinite factor, a subgroup of the maximal abelian factor of $(X, \mathcal{A}, \mu, T)$. The proof of the characteristic factor result above, due to Furstenberg [9], utilizes the characteristic factor for the three-term result. We are indebted to M. Lacey for bringing Furstenberg's theorems to our attention. However, a notable fact is that our proof for the bilinear Hilbert transform along a monimial curve does not have to go through the tri-linear Hilbert transform. The proof provided in this article heavily relies on the concept of "quadratic" uniformity and some kind of "quadratic" Fourier analysis, initiated by Gowers [10]. And perhaps this is a starting point to understand the tri-linear Hilbert transform.

Another prominent theme is the relation of the bilinear Hilbert transforms along curves and the multilinear oscillatory integrals. The bilinear Hilbert transforms along

curves are closely associated to the multilinear oscillatory integrals of the following type.

$$(1.2) \qquad \Lambda_\lambda(f_1, f_2, f_3) = \int_{\mathbf{B}} f_1(\mathbf{x} \cdot \mathbf{v}_1) f_2(\mathbf{x} \cdot \mathbf{v}_2) f_3(\mathbf{x} \cdot \mathbf{v}_3) e^{i\lambda\varphi(\mathbf{x})} d\mathbf{x},$$

where $\mathbf{B}$ is a unit ball in $\mathbb{R}^3$, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are vectors in $\mathbb{R}^3$, and the phase function φ satisfies a non-degenerate condition

$$(1.3) \qquad \left| \prod_{j=1}^{3} \left(\nabla \cdot \mathbf{v}_j^\perp \right) \varphi(\mathbf{x}) \right| \geq 1.$$

Here $\mathbf{v}_j^\perp$'s are unit vectors orthogonal to $\mathbf{v}_j$'s respectively. For a polynomial phase φ with the non-degenerate condition (1.3), it was proved in [4] that

$$(1.4) \qquad |\Lambda_\lambda(f_1, f_2, f_3)| \leq C(1 + |\lambda|)^{-\varepsilon} \prod_{j=1}^{3} \|f_j\|_\infty$$

holds for some positive number ε. For the particular vectors $\mathbf{v}_j$'s and the non-degenerate phase φ encountered in our problem, an estimate similar to (1.4) still holds. However, one of the main difficulties arises from the falsity of L^2 decay estimates for the trilinear form Λ_λ. In order to overcome this difficulty, we end up introducing the "quadratic" uniformity, which plays a role of a "bridge" connecting two spaces L^2 and L^∞.

The method used in this paper essentially works for those curves on nilpotent groups. It is possible to extend Theorem 1.1 to the general setting of nilpotent Lie groups. But we will not pursue this in this article. There are some related questions one can pose. Besides the generalisation to the more general curves, it is natural to ask the corresponding problems in higher dimensional cases and/or in multi-linear cases. For instance, in the tri-linear case, one can consider

$$(1.5) \qquad T(f_1, f_2, f_3)(x) = p.v. \int f_1(x + t) f_2(x + p_1(t)) f_3(x + p_2(t)) \frac{dt}{t}.$$

Here p_1, p_2 are polynomials of t. The investigation of such problems will be discussed in subsequent papers.

2. A LEMMA AND A COUNTEREXAMPLE

Let ρ be a Schwartz function supported in the union of two intervals $[-2, -1/2]$ and $[1/2, 2]$.

Lemma 2.1. *Let P be a real polynomial with degree $d \geq 2$. And let $2 \leq n \leq d$. Suppose that the n-th order derivative of P, $P^{(n)}$, does not vanish. Let $T(f, g)(x) =*

$\int f(x-t)g(x-P(t))\rho(t)dt$. *Then T is bounded from $L^p \times L^q$ to L^r for $p, q > 1$,* $r > \frac{n-1}{n}$ *and* $1/p + 1/q = 1/r$.

Proof. We may without loss of generality restrict x, hence likewise the supports of f, g, to fixed bounded intervals whose sizes depend on the coefficients of the polynomial P. This is possible because of the restriction $|t| \le 2$ in the integral. Let us restrict x in a bounded interval I_P. It is obvious that T is bounded uniformly from $L^\infty \times L^\infty$ to L^∞ and from $L^p \times L^{p'}$ to L^1 for $1 \le p \le \infty$ and $1/p + 1/p' = 1$. When $P'(t) \ne 1$ in $1/2 \le |t| \le 2$, then the boundedness from $L^1 \times L^1$ to L^1 can be obtained immediately by changing variable $u = x - t$ and $v = x - P(t)$ since the Jocobian $\frac{\partial(u,v)}{\partial(x,t)} = 1 - P'(t)$. Thus T is bounded from $L^1 \times L^1$ to $L^{1/2}$ since x is restricted to a bounded interval I_P and then the lemma follows by interpolation. When there is a real solution in $1/2 \le |t| \le 2$ to the equation $P'(t) = 1$, the trouble happens at a neighborhood of t_0, where $t_0 \in \{t : 1/2 \le |t| \le 2\}$ is the real solution to $P'(t) = 1$. There are at most $d - 1$ real solutions to the equation $P'(t) - 1 = 0$. Thus we only need to consider a small neighborhood containing only one real solution t_0 to $P'(t) = 1$. Let $I(t_0)$ be a small neighborhood of t_0 which contains only one real solution to $P'(t) - 1 = 0$. We should prove that

$$(2.1) \qquad \int_{I_P} \left| \int_{I(t_0)} f(x-t)g(x-P(t))\rho(t)dt \right|^r dx \le C_P \|f\|_p^r \|g\|_q^r,$$

for $p > 1, q > 1$ and $r > (n-1)/n$ with $1/p + 1/q = 1/r$. Let ρ_0 be a suitable bump function supported in $1/2 \le |t| \le 2$ such that $\sum_j \rho_0(2^j t) = 1$. To get (2.1), it suffices to prove that there is a positive ε such that

$$(2.2) \qquad \int_{I_P} \left| \int_{I(t_0)} f(x-t)g(x-P(t))\rho(t)\rho_0(2^j(t-t_0))dt \right|^r dx \le C2^{-\varepsilon j}\|f\|_p^r \|g\|_q^r,$$

for all large j, $p > 1, q > 1$ and $r > (n-1)/n$ with $1/p + 1/q = 1/r$, since (2.1) follows by summing for all possible $j \ge 1$. By a translation argument we need to show that

$$(2.3) \qquad \int_{I_P} \left| \int f(x-t)g(x-P_1(t))\rho_0(2^j t)dt \right|^r dx \le C2^{-\varepsilon j}\|f\|_p^r \|g\|_q^r,$$

for all large j, $p > 1, q > 1$ and $r > (n-1)/n$ with $1/p + 1/q = 1/r$, where P_1 is a polynomial of degree d defined by $P_1(t) = P(t + t_0) - P(t_0)$. It is clear that $P_1'(0) = 1$ and $P_1^{(n)} \ne 0$. When $|t| \le 2^{-j+1}$, $|P_1(t)| \le C_P 2^{-j}$ for some constant $C_P \ge 1$ depending on the coefficients of P. Let $I_P = [a_P, b_P]$ and A_N be defined by

$$A_N = [a_P + N C_P 2^{-j}, a_P + (N+1)C_P 2^{-j}] \text{ for } N = -1, \cdots, \frac{(b_P - a_P) \cdot 2^j}{C_P}.$$

Notice that for a fixed $x \in I_P$, $x - t, x - P_1(t)$ is in $A_{N-1} \cup A_N \cup A_{N+1}$ for some N. So we can restrict x in one of A_N's. Now let $T_N(f,g)(x) = 1_{A_N}(x) \int f(x-t)g(x-P_1(t))\rho_0(2^j t)dt$. Due to the restriction of x, we only need to show that

$$(2.4) \qquad \|T_N(f,g)\|_r^r \le C2^{-\varepsilon j}\|f\|_p^r \|g\|_q^r$$

for all large $j \geq 1$, $p > 1, q > 1$ and $r > (n-1)/n$ with $1/p + 1/q = 1/r$, where $f_N = f1_{A_N}$, $g_N = g1_{A_N}$ and C is independent of N.

By inserting absolute values throughout we get T_N maps $L^p \times L^q$ to L^r with a bound $C2^{-j}$ uniform in N, whenever $(1/p, 1/q, 1/r)$ belongs to the closed convex hull of the points $(1,0,1)$, $(0,1,1)$ and $(0,0,0)$. Observe that $P_1'(t) = 1 + \sum_{k=2}^{d-1} \frac{P_1^{(k)}(0)}{(k-1)!} t^{k-1}$ since $P_1'(0) = 1$. By $P_1^{(n)}(0) \neq 0$ and applying Cauchy-Schwarz inequality, we obtain for all j large enough,

$$\int \left|T_N(f,g)(x)\right|^{1/2} dx$$

$$\leq C_P 2^{-j/2} \|T_N(f,g)\|_1^{1/2}$$

$$\leq C_P 2^{-j/2} 2^{(n-1)j/2} \|f\|_1^{1/2} \|g\|_1^{1/2} = C_P 2^{(n-2)j/2} \|f\|_1^{1/2} \|g\|_1^{1/2}.$$

Hence an interpolation then yields a bound $C2^{-\varepsilon j}$ for all triples of reciprocal exponents within the convex hull of $(1, \frac{1}{n-1}, \frac{n}{n-1})$, $(\frac{1}{n-1}, 1, \frac{n}{n-1})$, $(1,0,1)$, $(0,1,1)$ and $(0,0,0)$. This finishes the proof of (2.4). Therefore we complete the proof of Lemma 2.1 □

Notice that if P is a monomial t^d, then the lower bound for r in Lemma 2.1 can be improved to $1/2$. This is because $P_1(t) = P(t + t_0) - P(t_0) = (t + t_0)^d - t_0^d$ has nonvanishing $P_1^{(2)}(0)$ when $1/2 \leq |t_0| \leq 1$. We now give a counterexample to indicate that the lower bound $(n-1)/n$ for r is sharp in Lemma 2.1.

Proposition 2.1. *Let d, n be integers such that $d \geq 2$ and $2 \leq n \leq d$. There is a real polynomial Q of degree $d \geq 2$ whose n-th order derivative does not vanish such that T_Q is unbounded from $L^p \times L^q$ to L^r for all $p, q > 1$ and $r < (n-1)/n$ with $1/p+1/q = 1/r$, where T_Q is the bilinear operator defined by $T_Q(f,g)(x) = \int f(x-t)g(x - Q(t))\rho(t)dt$.*

Proof. Let A be a very large number. We define $Q(t)$ by

$$(2.5) \qquad Q(t) = \frac{1}{Ad!}(t-1)^d + \frac{1}{An!}(t-1)^n + (t-1).$$

It is sufficient to prove that if T_Q is bounded from $L^p \times L^q$ to L^r for some $p, q > 1$ and $1/r = 1/p + 1/q$, then $r \geq (n-1)/n$. Suppose there is a constant C such that $\|T_Q(f,g)\|_r \leq C\|f\|_p\|g\|_q$ for all $f \in L^p$ and $g \in L^q$. Let δ be a small positive number. And let $f_\delta = 1_{[0,2^n\delta]}$ and $g_\delta = 1_{[1-\delta,1]}$. Let D_1 be the intersection point of the curves $x = Q(t) + 1$ and $x = t + 2^n\delta$ in tx-plane with $t > 1$, and let D_2 be the intersection point of the curves $x = Q(t)+1-\delta$ and $x = t$ in tx-plane with $t > 1$. Let $D_1 = (t_1, x_1)$ and $D_2 = (t_2, x_2)$. Then

$$1 + 2^{1-1/n}(An!)^{1/n}\delta^{1/n} \leq t_1 \leq 1 + 2(An!)^{1/n}\delta^{1/n} \text{ and}$$

$$1 + 2^{-1/n}(An!)^{1/n}\delta^{1/n} \leq t_2 \leq 1 + (An!)^{1/n}\delta^{1/n}.$$

Thus we have

$$1 + 2^{1-1/n}(An!)^{1/n}\delta^{1/n} + 2^n\delta \leq x_1 \leq 1 + 2(An!)^{1/n}\delta^{1/n} + 2^n\delta \text{ and}$$

$$1 + 2^{-1/n}(An!)^{1/n}\delta^{1/n} \leq x_2 \leq 1 + (An!)^{1/n}\delta^{1/n}.$$

When A is large and δ is small, any horizontal line between line $x = x_1$ and line $x = x_2$ has a line segment of length $\delta/2$ staying within the region bounded by curves $x = t$, $x = Q(x) + 1 - \delta$, $x = t + 2^n\delta$ and $x = Q(t) + 1$. Hence, we have

$$(2.6) \qquad \|T_Q(f_\delta, g_\delta)\|_r^r \geq (\delta/2)^r (An!)^{1/n} \delta^{1/n} / 100 \,.$$

By the boundedness of T_Q, we have

$$\|T_Q(f_\delta, g_\delta)\|_r^r \leq C^r (2^n\delta)^{r/p} \delta^{r/q} = C^r 2^{nr/p} \delta \,.$$

By (2.6) we have

$$(2.7) \qquad \delta^r \leq \frac{1002^{r+nr/p} C^r}{(An!)^{1/n}} \delta^{\frac{n-1}{n}} \,.$$

Since A can be chosen to be a very large number and δ can be very small, (2.7) implies $r \geq \frac{n-1}{n}$, which completes the proof of Lemma 2.1. $\qquad\qquad\square$

3. A Decomposition

Let ρ_1 be a standard bump function supported on $[1/2, 2]$. And let

$$\rho(t) = \rho_1(t)\mathbf{1}_{\{t>0\}} - \rho_1(-t)\mathbf{1}_{\{t<0\}} \,.$$

It is clear that ρ is an odd function. To obtain the L^r estimates for H_Γ, it is sufficient to get L^r estimates for T_Γ defined by $T_\Gamma = \sum_{j\in\mathbb{Z}} T_{\Gamma,j}$, where $T_{\Gamma,j}$ is

$$(3.1) \qquad T_{\Gamma,j}(f, g)(x) = \int f(x - t)g(x - t^d) 2^j \rho(2^j t) dt \,.$$

Let L be a large positive number (larger than 2^{100}). By Lemma 2.1, we have that if $|j| \leq L$, $\|T_{\Gamma,j}(f, g)\|_r \leq C_L \|f\|_p \|g\|_q$ for all $p, q > 1$ and $1/p + 1/q = 1/r$, where the operator norm C_L depends on the upper bound L. Hence in the following we only need to consider the case when $|j| > L$. In fact we prove the following theorem.

Theorem 3.1. *Let $T_{\Gamma,j}$ be defined as in (3.1). Then the bilinear operator $T_L = \sum_{j\in\mathbb{Z}:|j|>L} T_{\Gamma,j}$ is bounded from $L^2 \times L^2$ to L^1.*

Clearly Theorem 1.1 follows by Theorem 3.1 and Lemma 2.1. The rest part of the article is devoted to a proof of Theorem 3.1.

We begin the proof of Theorem 3.1 by constructing an appropriate decomposition of the operator $T_{\Gamma,j}$. This is done by an analysis of the bilinear symbol associated with the operator.

Expressing $T_{\Gamma,j}$ in dual frequency variables, we have

$$T_{\Gamma,j}(f, g)(x) = \int\int \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)x}\mathfrak{m}_j(\xi, \eta)d\xi d\eta \,,$$

where the symbol $\mathfrak{m}_j$ is defined by

$$(3.2) \qquad \mathfrak{m}_j(\xi, \eta) = \int \rho(t) e^{-2\pi i\left(2^{-j}\xi t + 2^{-dj}\eta t^d\right)} dt\,.$$

First we introduce a resolution of the identity. Let Θ be a Schwarz function supported on $(-1, 1)$ such that $\Theta(\xi) = 1$ if $|\xi| \leq 1/2$. Set Φ to be a Schwarz function satisfying

$$\widehat{\Phi}(\xi) = \Theta(\xi/2) - \Theta(\xi)\,.$$

Then Φ is a Schwarz function such that $\widehat{\Phi}$ is supported on $\{\xi : 1/2 < |\xi| < 2\}$ and

$$(3.3) \qquad \sum_{m \in \mathbb{Z}} \widehat{\Phi}\left(\frac{\xi}{2^m}\right) = 1 \text{ for all } \xi \in \mathbb{R}\backslash\{0\}\,,$$

and for any $m_0 \in \mathbb{Z}$,

$$(3.4) \qquad \widehat{\Phi}_{m_0}(\xi) := \sum_{m=-\infty}^{m_0} \widehat{\Phi}\left(\frac{\xi}{2^m}\right) = \Theta\left(\frac{\xi}{2^{m_0+1}}\right)\,,$$

which is a bump function supported on $(-2^{m_0+1}, 2^{m_0+1})$.

From (3.3), we can decompose $T_{\Gamma,j}$ into two parts: $T_{\Gamma,j,1}$ and $T_{\Gamma,j,2}$, where $T_{\Gamma,j,1}$ is given by

$$(3.5) \qquad \sum_{m \in \mathbb{Z}} \sum_{\substack{m' \in \mathbb{Z}: \\ |m'-m| > 10^d}} \iint \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)x} \widehat{\Phi}\left(\frac{2^{-j}\xi}{2^m}\right) \widehat{\Phi}\left(\frac{2^{-dj}\eta}{2^{m'}}\right) \mathfrak{m}_j(\xi, \eta) d\xi d\eta\,,$$

and $T_{\Gamma,j,2}$ is defined by

$$(3.6) \qquad \sum_{m \in \mathbb{Z}} \sum_{\substack{m' \in \mathbb{Z}: \\ |m'-m| \leq 10^d}} \iint \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)x} \widehat{\Phi}\left(\frac{2^{-j}\xi}{2^m}\right) \widehat{\Phi}\left(\frac{2^{-dj}\eta}{2^{m'}}\right) \mathfrak{m}_j(\xi, \eta) d\xi d\eta\,.$$

Define $\mathfrak{m}_d$ by

$$(3.7) \qquad \mathfrak{m}_d(\xi, \eta) = \int \rho(t) e^{-2\pi i(\xi t + \eta t^d)} dt\,.$$

Clearly $\mathfrak{m}_j(\xi, \eta) = \mathfrak{m}_d(2^{-j}\xi, 2^{-dj}\eta)$. In $T_{\Gamma,j,1}$, the phase function $\phi_{\xi,\eta}(t) = \xi t + \eta t^d$ does not have any critical point in a neighborhood of the support of ρ, and therefore a very rapid decay can be obtained by integration by parts so that we can show that $\sum_j T_{\Gamma,j,1}$ is essentially a finite sum of paraproducts (see Section 4). A critical point of the phase function may occur in $T_{\Gamma,j,2}$ and therefore the method of stationary phase must be brought to bear in this case, exploiting in particular the oscillatory term. This case requires the most extensive analysis.

Notice that there are only finitely many m' if m is fixed in (3.6). Without loss of generality, we can assume $m' = m$. Then in order to get the L^r estimates for $\sum_j T_{\Gamma,j,2}$, it suffices to prove the L^r boundedness of $\sum_m T_m$, where T_m's are defined by

$$(3.8) \qquad T_m(f, g)(x) = \sum_{|j| > L} \iint \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i(\xi+\eta)x} \widehat{\Phi}\left(\frac{2^{-j}\xi}{2^m}\right) \widehat{\Phi}\left(\frac{2^{-dj}\eta}{2^m}\right) \mathfrak{m}_j(\xi, \eta) d\xi d\eta\,.$$
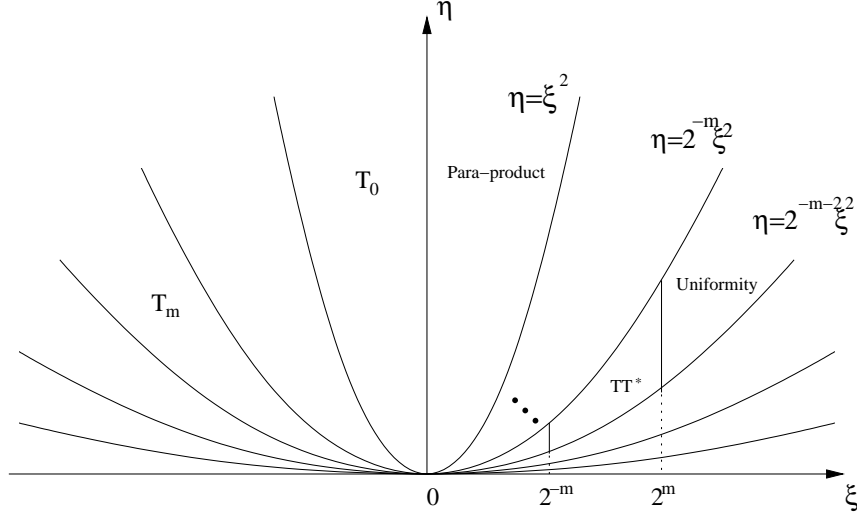
FIGURE 1. The decomposition of (ξ, η)-plane for $\sum_m T_m$ when $d = 2$

It can be proved that $T_0 = \sum_{m \leq 0} T_m$ is equal to $\sum_{m \leq 0} O(2^{m/2})\Pi_m$, where Π_m is a paraproduct studied in Theorem 4.1. This can be done by Fourier series and the cancellation condition of ρ and thus T_0 is essentially a paraproduct. We omit the details for it since it is exactly same as those in Section 4 for the case $\sum_j T_{\Gamma,j,1}$. Therefore, the most difficult term is $\sum_{m \geq 1} T_m$. For this term, we have the following theorem.

Theorem 3.2. *Let T_m be a bilinear operator defined as in (3.8). Then there exists a constant C such that*

$$(3.9) \qquad \left\| \sum_{m \geq 1} T_m(f, g) \right\|_1 \leq C \|f\|_2 \|g\|_2$$

holds for all $f, g \in L^2$.

A delicate analysis is required for proving this theorem. We will prove it in Subsection 5.1. Theorem 3.1 follows from Theorem 3.2 and the boundedness of $\sum_j T_{\Gamma,j,1}$. The rest of the article is organized as follows. In Section 4, the L^r-boundedness will be established for $\sum_j T_{\Gamma,j,1}$. Some crucial bilinear restriction estimates will appear in Section 5 and as a consequence Theorem 3.2 follows. Sections 6-11 are devoted to a proof of the bilinear restriction estimates.

4. PARAPRODUCTS AND UNIFORM ESTIMATES

In this section we prove that $\sum_j T_{\Gamma,j,1}$ is essentially a finite sum of certain paraproducts bounded from $L^p \times L^q$ to L^r.

First let us introduce the paraproduct encountered in our problem. Let $j \in \mathbb{Z}$, L_1, L_2 be positive integers and M_1, M_2 be integers.

$$\omega_{1,j} = [2^{L_1 j + M_1}/2, 2 \cdot 2^{L_1 j + M_1}]$$

and

$$\omega_{2,j} = \left[-2^{L_2 j + M_2}, 2^{L_2 j + M_2}\right].$$

Let Φ_1 be a Schwartz function whose Fourier transform is a standard bump function supported on a small neighborhood of $[1/2, 2]$ or $[-2, -1/2]$, and Φ_2 be a Schwartz function whose Fourier transform is a standard bump function supported on $[-1, 1]$ and $\widehat{\Phi}_2(0) = 1$. For $\ell \in \{1, 2\}$ and $n_1, n_2 \in \mathbb{Z}$, define Φ_{ℓ,j,n_ℓ} by

$$\widehat{\Phi}_{\ell,j,n_\ell}(\xi) = \left(e^{2\pi i n_\ell(\cdot)}\widehat{\Phi}_\ell(\cdot)\right)\left(\frac{\xi}{2^{L_\ell j + M_\ell}}\right).$$

It is clear that $\widehat{\Phi}_{\ell,j,n_\ell}$ is supported on $\omega_{\ell,j}$. For locally integrable functions f_ℓ's, we define $f_{\ell,j}$'s by

$$f_{\ell,j,n_\ell}(x) = f_\ell * \Phi_{\ell,j,n_\ell}(x).$$

We now define a paraproduct to be

(4.10) $$\Pi_{L_1,L_2,M_1,M_2,n_1,n_2}(f_1, f_2)(x) = \sum_{j \in \mathbb{Z}} \prod_{\ell=1}^{2} f_{\ell,j,n_\ell}(x).$$

For this paraproduct, we have the following uniform estimates.

Theorem 4.1. *For any $p_1 > 1$, $p_2 > 1$ with $1/p_1 + 1/p_2 = 1/r$, there exists a constant C independent of M_1, M_2, n_1, n_2 such that*

(4.11) $$\left\|\Pi_{L_1,L_2,M_1,M_2,n_1,n_2}(f_1, f_2)\right\|_r \leq C\left(1 + |n_1|\right)^{10}\left(1 + |n_2|\right)^{10}\|f_1\|_{p_1}\|f_2\|_{p_2},$$

for all $f_1 \in L^{p_1}$ and $f_2 \in L^{p_2}$.

The $r > 1$ case can be handled by a telescoping argument. The $r < 1$ case is more complicated and it requires a time-frequency analysis. A proof of Theorem 4.1 can be found in [15]. The constant C in Theorem 4.1 may depend on L_1, L_2. It is easy to see that C is $O(\max\{2^{L_1}, 2^{L_2}\})$. It is possible to get a much better upper bound such as $O\left(\log(1 + \max\{L_2/L_1, L_1/L_2\})\right)$ by tracking the constants carefully in the proof in [15]. But we do not need the sharp constant in this article. The independence on M_1, M_2 is the most important issue here.

We now return to $\sum_j T_{\Gamma,j,1}$. This sum can be written as $T_{L,1} + T_{L,2}$, where $T_{L,1}$ is a bilinear operator defined by

$$\sum_{|j|>L} \sum_{m \in \mathbb{Z}} \sum_{\substack{m' \in \mathbb{Z} \\ m' < m - 10^d}} \int \int \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)x}\widehat{\Phi}\left(\frac{2^{-j}\xi}{2^m}\right)\widehat{\Phi}\left(\frac{2^{-dj}\eta}{2^{m'}}\right)\mathfrak{m}_j(\xi, \eta)d\xi d\eta,$$

and $T_{L,2}$ is a bilinear operator given by

$$\sum_{|j|>L} \sum_{m' \in \mathbb{Z}} \sum_{\substack{m \in \mathbb{Z} \\ m < m' - 10^d}} \int \int \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)x}\widehat{\Phi}\left(\frac{2^{-j}\xi}{2^m}\right)\widehat{\Phi}\left(\frac{2^{-dj}\eta}{2^{m'}}\right)\mathfrak{m}_j(\xi, \eta)d\xi d\eta.$$

Case $T_{L,1}$. We now prove that $T_{L,2}$ can be reduced to the paraproducts studied in [15]. Indeed, if $|\xi| > 5^d|\eta|$, then let $\tilde{\mathfrak{m}}$ be

$$\tilde{\mathfrak{m}}(\xi, \eta) = \mathfrak{m}_d(\xi, \eta) - \int \rho(t)e^{-2\pi i\xi t}dt\,.$$

Clearly, if $|\xi| > 5^d|\eta|$,

(4.12) $$\left|D^\alpha\tilde{\mathfrak{m}}(\xi, \eta)\right| \leq \frac{C_{\alpha,N}}{(1 + \sqrt{|\xi|^2 + |\eta|^2})^N} \text{ for all } N \in \{0\} \cup \mathbb{N}.$$

and

(4.13) $$\left|\tilde{\mathfrak{m}}(\xi, \eta)\right| \leq C_d \min\{1, |\eta|\}\,.$$

Thus if $m' \leq m - 10^d$, then we have

$$\tilde{\mathfrak{m}}(2^m\xi, 2^{m'}\eta)\widehat{\Phi}(\xi)\widehat{\Phi}(\eta) = \sum_{n_1,n_2\in\mathbb{Z}} C^{(1)}_{n_1,n_2}e^{2\pi i(n_1\xi + n_2\eta)}\,,$$

where $C^{(1)}_{n_1,n_2}$ is the Fourier coefficient. From (4.12) and (4.13), we have

(4.14) $$|C^{(1)}_{n_1,n_2}| \leq \frac{C\min\{1, 2^{m'/2}\}}{(1 + \sqrt{n_1^2 + n_2^2})^N(1 + 2^m + 2^{m'})^N} \text{ for all } N\,.$$

And notice that ρ is an odd function, then we have

$$\left(\int \rho(t)e^{-2\pi i2^m\xi t}dt\right)\widehat{\Phi}(\xi) = \sum_n C^{(1)}_n e^{2\pi in\xi}\,,$$

where the Fourier coefficient $C^{(1)}_n$ satisfies

(4.15) $$\left|C^{(1)}_n\right| \leq \frac{C_N\min\{2^{m/2}, 1\}}{(1 + |n| + 2^m)^N} \text{ for all } N\,.$$

Thus $T_{L,1}$ equals to a sum of paraproducts
(4.16)
$$\sum_{j<-B}\sum_{m\in\mathbb{Z}}\sum_{\substack{m'\in\mathbb{Z}\\m'\leq m-10^d}}\left(\sum_{n_1,n_2\in\mathbb{Z}} C^{(1)}_{n_1,n_2}f_{j,m,n_1}(x)g_{j,m',n_2}(x) + \sum_{n\in\mathbb{Z}} C^{(1)}_n f_{j,m,n}(x)g_{j,m',0}(x)\right),$$

where $f_{j,m,n}$ is a function whose Fourier transform is

$$\widehat{f}_{j,m,n}(\xi) = \widehat{f}(\xi)\widehat{\Phi}\big(2^{-j-m}\xi\big)e^{2\pi in2^{-j-m}\xi}\,,$$

$g_{j,m',n}$ is a function whose Fourier transform is

$$\widehat{g}_{j,m',n}(\eta) = \widehat{g}(\eta)\widehat{\Phi}\big(2^{-dj-m'}\eta\big)e^{2\pi in2^{-dj-m'}\eta}\,.$$

Remark 4.1. *Actually, in the definition of $f_{j,m,n}$ and $g_{j,m',n}$, $\widehat{\Phi}$ should be a Schwartz function supported in some neighborhood of $\widehat{\Phi}$ and it is identically equal to 1 on the support of $\widehat{\Phi}$. We abuse the notions here. But it does no harm to us since the propety of the function does not change significantly.*

Notice that the Fourier transform of $\sum_{m'<m-10^d} g_{j,m',0}$ is a function supported in the interval $I_{j,m,2} = [-2^{dj+m}, 2^{dj+m}]$. We denote $\sum_{m'<m-10^d} g_{j,m',0}$ by $g_{j,m}$. Thus by the definition of Φ, $\widehat{g}_{j,m} = \widehat{g}\widehat{\Phi}_m$ where Φ_m is a standard bump function supported in $I_{j,m,2}$. Let $I_{j,m,1} = [2^{j+m}/2, 2^{j+m+1}]$. Then $\widehat{f}_{j,m,n}$ is supported in $I_{j,m,1}$. Hence due to the fast decay of the Fourier coefficients (4.14) and (4.15), we actually run into two paraproducts in this case. One of them is

$$(4.17) \qquad \Pi_{L,1}^{(1)}(f,g)(x) = \sum_j f_{j,m,n_1}(x)g_{j,m',n_2}(x)\,.$$

The other is

$$(4.18) \qquad \Pi_{L,1}^{(2)}(f,g)(x) = \sum_j f_{j,m,n}(x)g_{j,m}(x)\,.$$

The L^r estimates of these paraproducts follow from Theorem 4.1. In fact, for all $p, q > 1$ and $1/p + 1/q = 1/r$, applying Theorem 4.1, we obtain the $L^p \times L^q \to L^r$ estimates uniformly in m, m' with the operator norms $O((1 + |n_1| + |n_2|)^{20})$ and $O((1 + |n|)^{10})$ for $\Pi_{L,1}^{(1)}$ and $\Pi_{L,1}^{(2)}$ respectively. The fast decay estimates of the Fourier coefficients (4.14) and (4.15) then allow us to conclude the desired L^r boundedness of $T_{L,1}$.

Case $T_{L,2}$. This case is similar to the case $T_{L,1}$. In fact, if $|\eta| > 5^d|\xi|$, then let $\tilde{\mathfrak{m}}$ be

$$\tilde{\mathfrak{m}}(\xi,\eta) = \mathfrak{m}_d(\xi,\eta) - \int \rho(t)e^{-2\pi i\eta t^d}\,dt\,.$$

Clearly, if $|\eta| > 5^d|\xi|$,

$$(4.19) \qquad \left|D^\alpha\tilde{\mathfrak{m}}(\xi,\eta)\right| \leq \frac{C_{\alpha,N}}{(1 + \sqrt{|\xi|^2 + |\eta|^2})^N} \quad \text{for all } N \in \{0\} \cup \mathbb{N}\,.$$

and

$$(4.20) \qquad \left|\tilde{\mathfrak{m}}(\xi,\eta)\right| \leq C_d \min\{1, |\xi|\}\,.$$

Thus if $m \leq m' - 10^d$, then we have

$$\tilde{\mathfrak{m}}(2^m\xi, 2^{m'}\eta)\widehat{\Phi}(\xi)\widehat{\Phi}(\eta) = \sum_{n_1,n_2\in\mathbb{Z}} C_{n_1,n_2}^{(2)}e^{2\pi i(n_1\xi+n_2\eta)}\,,$$

where $C_{n_1,n_2}^{(2)}$ is the Fourier coefficient. From (4.19) and (4.20), we have

$$(4.21) \qquad |C_{n_1,n_2}^{(2)}| \leq \frac{C\min\{1, 2^{m/2}\}}{(1 + \sqrt{n_1^2 + n_2^2})^N(1 + 2^m + 2^{m'})^N} \quad \text{for all } N\,.$$

And notice that ρ is an odd function, then we have

$$\left(\int \rho(t)e^{-2\pi i2^{m'}\eta t^d}\,dt\right)\widehat{\Phi}(\xi) = \sum_n C_n^{(2)}e^{2\pi in\eta}\,,$$

where the Fourier coefficient $C_n^{(2)}$ satisfies

$$(4.22) \qquad \left|C_n^{(2)}\right| \leq \frac{C_N \min\{2^{m'/2}, 1\}}{(1 + |n| + 2^{m'})^N} \text{ for all } N.$$

Thus $T_{L,2}$ equals to a sum of paraproducts
$$(4.23)$$
$$\sum_{j<-B} \sum_{m'\in\mathbb{Z}} \sum_{\substack{m\in\mathbb{Z} \\ m\leq m'-10^d}} \left(\sum_{n_1,n_2\in\mathbb{Z}} C_{n_1,n_2}^{(2)} f_{j,m,n_1}(x) g_{j,m',n_2}(x) + \sum_{n\in\mathbb{Z}} C_n^{(2)} f_{j,m,0}(x) g_{j,m',n}(x) \right).$$

Observe that the Fourier transform of $\sum_{m<m'-10^d} f_{j,m,0}$ is a function supported in the interval $I_{j,m',1} = [-2^{j+m'}, 2^{j+m'}]$. We denote $\sum_{m<m'-10^d} f_{j,m,0}$ by $f_{j,m'}$. Thus by the definition of Φ, $\widehat{f}_{j,m'} = \widehat{f}\widetilde{\Phi}_{m'}$ where $\Phi_{m'}$ is a standard bump function supported in $I_{j,m',1}$. Let $I_{j,m',2} = [2^{j+m'}/2, 2^{j+m'+1}]$. Then $\widehat{g}_{j,m',n}$ is supported in $I_{j,m',2}$. Hence due to the rapid decay of the Fourier coefficients (4.21) and (4.22), we actually encounter two paraproducts in this case. One of them is

$$(4.24) \qquad \Pi_{L,2}^{(1)}(f,g)(x) = \sum_j f_{j,m,n_1}(x) g_{j,m',n_2}(x).$$

The other is

$$(4.25) \qquad \Pi_{L,2}^{(2)}(f,g)(x) = \sum_j f_{j,m'}(x) g_{j,m',n}(x).$$

Applying Theorem 4.1, for all $p, q > 1$ and $1/p + 1/q = 1/r$, we have the $L^p \times L^q \to L^r$ estimates uniformly in m, m' with the operator norms $O((1 + |n_1| + |n_2|)^{20})$ and $O((1 + |n|)^{10})$ for $\Pi_{L,2}^{(1)}$ and $\Pi_{L,2}^{(2)}$ respectively. Then the desired L^r estimates of $T_{L,2}$ follow due to the fast decay of the Fourier coefficients (4.21) and (4.22).

5. Bilinear Fourier Restriction Estimates

Let $d \geq 2, m \geq 0, j \in \mathbb{Z}$. We define a bilinear Fourier restriction operator of f, g by

$$(5.1) \quad \mathcal{B}_{j,m}(f,g)(x) = 2^{-(d-1)j/2} \int_{\mathbb{R}} R_\Phi f(2^{-(d-1)j}x - 2^m t) R_\Phi g(x - 2^m t^d) \rho(t)dt \text{ if } j \geq 0$$

and

$$(5.2) \quad \mathcal{B}_{j,m}(f,g)(x) = 2^{(d-1)j/2} \int_{\mathbb{R}} R_\Phi f(x - 2^m t) R_\Phi g(2^{(d-1)j}x - 2^m t^d) \rho(t)dt \text{ if } j < 0,$$

where $R_\Phi f$ and $R_\Phi g$ are the Fourier (smooth) restrictions of f, g on the support of $\widehat{\Phi}$ respectively. More precisely, $R_\Phi f, R_\Phi g$ are given by

$$(5.3) \qquad \widehat{R_\Phi f}(\xi) = \widehat{f}(\xi)\widehat{\Phi}(\xi)$$

$$(5.4) \qquad \widehat{R_\Phi g}(\xi) = \widehat{g}(\xi)\widehat{\Phi}(\xi)$$

By inserting absolute values throughout and applying Cauchy-Schwarz inequality, the boundedness of $\mathcal{B}_{j,m}$ from $L^2 \times L^2$ to L^1 follows immediately. Moreover, since the Fourier transform of f, g are restricted on the support of $\widehat{\Phi}$, we actually can improve the estimate. Let us state the improved estimates by the following theorems, which are of independent interest.

Theorem 5.1. *Let $d \geq 2$ and $\mathcal{B}_{j,m}$ be defined as in (5.1) and (5.2). If $L \leq |j| \leq m/(d-1)$, then there exists a constant C independent of j, m such that*

$$(5.5) \qquad \left\| \mathcal{B}_{j,m}(f,g) \right\|_1 \leq C 2^{\frac{(d-1)|j|-m}{8}} \|f\|_2 \|g\|_2 \,,$$

holds for all $f, g \in L^2$.

Theorem 5.2. *Let $d \geq 2$ and $\mathcal{B}_{j,m}$ be defined as in (5.1) and (5.2). If $|j| \geq m/(d-1)$, then there exists a positive number ε_0 and a constant C independent of j, m such that*

$$(5.6) \qquad \left\| \mathcal{B}_{j,m}(f,g) \right\|_1 \leq C \max \left\{ 2^{\frac{m-(d-1)|j|}{3}}, 2^{-\varepsilon_0 m} \right\} \|f\|_2 \|g\|_2 \,,$$

holds for all $f, g \in L^2$.

The positive number ε_0 in Theorem 5.2 can be chosen to be $1/(8d)$. Theorem 5.1 can be proved by a TT^* method. However, the TT^* method fails when $|j| > m/(d-1)$. To obtain Theorem 5.2, we will employ a method related to the uniformity of functions.

Now we can see that Theorem 3.2 is a consequence of Theorem 5.1 and Theorem 5.2.

5.1. **Proof of Theorem 3.2.** Define a bilinear operator $T_{j,m}$ to be

$$(5.7) \qquad T_{j,m}(f,g)(x) = \iint \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)x}\widehat{\Phi}\left(\frac{2^{-j}\xi}{2^m}\right)\widehat{\Phi}\left(\frac{2^{-dj}\eta}{2^m}\right)\mathfrak{m}_j(\xi,\eta)d\xi d\eta \,.$$

Let $\gamma_{j,m}$ be defined by

$$(5.8) \qquad \gamma_{j,m} = \begin{cases} 2^{\frac{(d-1)|j|-m}{8}} & \text{if } |j| \leq \frac{m}{d-1} \\ \max \left\{ 2^{\frac{m-(d-1)|j|}{3}}, 2^{-\varepsilon_0 m} \right\} & \text{if } |j| \geq \frac{m}{d-1} \end{cases}$$

A rescaling argument, Theorem 5.1 and Theorem 5.2 yield

$$(5.9) \qquad \|T_{j,m}(f,g)\|_1 \leq C\gamma_{j,m}\|f\|_2\|g\|_2 \,.$$

Since $\sum_m T_m = \sum_m \sum_{j:|j|\geq L} T_{j,m}$, we obtain

$$(5.10) \qquad \left\| \sum_{m\geq 1} T_m(f,g) \right\|_1 \leq C \sum_{m\geq 1} \sum_{j:|j|\geq L} \gamma_{j,m} \|f_{j,m}\|_2 \|g_{j,m}\|_2 \,,$$

where

$$\widehat{f_{j,m}}(\xi) = \widehat{f}(\xi)\widehat{\Phi}\left(\frac{\xi}{2^{j+m}}\right) \,,$$

$$\widehat{g_{j,m}}(\eta) = \widehat{g}(\eta)\widehat{\Phi}\left(\frac{\eta}{2^{dj+m}}\right) \,.$$

Clearly the right hand side of (5.10) is bounded by $C\|f\|_2\|g\|_2$. Therefore we finish the proof of Theorem 3.2.

We now start to make some reductions first for proving Theorem 5.1 and Theorem 5.2.

5.2. Smooth Truncations. Let ϕ be a nonnegative Schwartz function such that $\widehat{\phi}$ is supported in $[-1/100, 1/100]$ and satisfies $\widehat{\phi}(0) = 1$. For any integer k, let $\phi_k(x) = 2^{-k}\phi(2^{-k}x)$. And for $n \in \mathbb{Z}$, let

$$I_{k,n} = [2^k n, 2^{k+1} n].$$

Denote the characteristic function of the set I by $\mathbf{1}_I$. We define

$$\mathbf{1}_{k,n}^*(x) = \mathbf{1}_{I_{k,n}} * \phi_k(x),$$

and

$$\mathbf{1}_{k,n}^{**}(x) = \int_{I_{k,n}} \frac{2^{-k}}{\left(1 + 2^{-k}|x - y|\right)^{200}} dy.$$

Here $\mathbf{1}_{k,n}^*$ and $\mathbf{1}_{k,n}^{**}$ can be considered as essentially $\mathbf{1}_{I_{k,n}}$. Clearly we have

$$(5.11) \qquad\qquad \sum_n \mathbf{1}_{k,n}^*(x) = 1$$

Lemma 5.1. *Let Φ_1 be a Schwartz function such that $\widehat{\Phi}_1(\xi) \equiv 1$ if $3/8 \le |\xi| \le 17/8$ and $\widehat{\Phi}_1$ is supported on $1/8 \le |\xi| \le 19/8$. And let $n \in \mathbb{Z}$ and define $\mathcal{B}_{j,m,n}$ by if $j > 0$, then*
(5.12)
$$\mathcal{B}_{j,m,n}(f,g)(x) = 2^{-(d-1)j/2}\int_{\mathbb{R}} R_{\Phi_1}f(2^{-(d-1)j}x - 2^m t)R_{\Phi_1}g(x - 2^m t^d)\rho(t)dt\,\mathbf{1}_{(d-1)j+m,n}^*(x),$$

if $j \le 0$, then
(5.13)
$$\mathcal{B}_{j,m,n}(f,g)(x) = 2^{(d-1)j/2}\int_{\mathbb{R}} R_{\Phi_1}f(x - 2^m t)R_{\Phi_1}g(2^{(d-1)j}x - 2^m t^d)\rho(t)dt\,\mathbf{1}_{(d-1)|j|+m,n}^*(x),$$

If there is a constant $C_{j,m}$ independent of n such that

$$(5.14) \qquad\qquad \|\mathcal{B}_{j,m,n}(f,g)\|_1 \le C_{j,m}\|f\|_2\|g\|_2$$

holds for all $f, g \in L^2$, then for any positive number ε,

$$(5.15) \qquad\qquad \|\mathcal{B}_{j,m}(f,g)\|_1 \le C_\varepsilon C_{j,m}^{1-\varepsilon}\|f\|_2\|g\|_2,$$

where C_ε is a constant depending on ε only.

Proof. Without loss of generality, assume that $C_{j,m} \le 1$. And we only prove the case $j > 0$. The case $j \le 0$ can be proved similarly and we omit the proof for this case. From (5.11), we can express $\langle \mathcal{B}_{j,m}(f,g), h \rangle$ as

$$\sum_{k_1}\sum_{k_2}\sum_n \Lambda_{k_1,k_2,n,j,m}(f,g,h).$$

Here $\Lambda_{k_1,k_2,n,j,m}(f,g,h)$ equals to

$$(5.16) \quad 2^{-\frac{(d-1)j}{2}} \iint F_{1,j,m,n,k_1}(2^{-(d-1)j}x - 2^m t) F_{2,j,m,n,k_2}(x - 2^m t^d) F_{3,j,m,n}(x)\,dxdt\,,$$

where

$$F_{1,j,m,n,k_1} = \mathbf{1}^*_{m,n+k_1} R_\Phi f\,,$$
$$F_{2,j,m,n,k_2} = \mathbf{1}^*_{(d-1)j+m,n+k_2} R_\Phi g\,,$$
$$F_{3,j,m,n} = \mathbf{1}^*_{(d-1)j+m,n} h\,.$$

Putting the absolute value throughout and utilizing the fast decay of $\mathbf{1}^*_{k,n}$, we estimate the sum of $\Lambda_{k_1,k_2,n,j,m}(f,g,h)$ for all (k_1,k_2,n)'s with $\max\{|k_1|,|k_2|\} \geq C_{j,m}^{-\varepsilon/2}$ by

$$2^{-\frac{(d-1)j}{2}} \sum_{\substack{(k_1,k_2,n) \\ \max\{|k_1|,|k_2|\} \geq C_{j,m}^{-\varepsilon/2}}} \iint \frac{\left|R_\Phi f(2^{-(d-1)j}x - 2^m t)\right|\left|R_\Phi g(x - 2^m t^d)\right||h(x)||\rho(t)|}{\left(1 + |t + k_1|\right)^N \left(1 + |2^{-(d-1)j}t^d + k_2|\right)^N}\,dxdt\,,$$

for all positive integers N. Since $|t| \sim 1$ when t is in the support of ρ, we dominate this sum by

$$(5.17) \qquad\qquad C_\varepsilon C_{j,m}\|f\|_2\|g\|_2\|h\|_\infty\,.$$

We now turn to sum $\Lambda_{k_1,k_2,n,j,m}(f,g,h)$ for all $|k_1| < C_{j,m}^{-\varepsilon/2}$ and $|k_2| < C_{j,m}^{-\varepsilon/2}$. Observe that when j,m are large, Fourier transforms of F_{1,j,m,n,k_1} and F_{2,j,m,n,k_2} are supported in $3/8 \leq |\xi| \leq 17/8$. Thus we have

$$\Lambda_{k_1,k_2,n,j,m}(f,g,h) = \langle \mathcal{B}_{j,m,n}(F_{1,j,m,n,k_1}, F_{2,j,m,n,k_2}), h \rangle\,.$$

And then (5.14) gives

$$\sum_{\substack{(k_1,k_2,n) \\ \max\{|k_1|,|k_2|\} < C_{j,m}^{-\varepsilon/2}}} \left|\Lambda_{k_1,k_2,n,j,m}(f,g,h)\right| \leq C_{j,m} \sum_{\substack{(k_1,k_2,n) \\ \max\{|k_1|,|k_2|\} < C_{j,m}^{-\varepsilon/2}}} \|F_{1,j,m,n,k_1}\|_2\|F_{2,j,m,n,k_2}\|_2\|h\|_\infty\,,$$

which is clearly bounded by

$$(5.18) \qquad\qquad C_{j,m}^{1-\varepsilon}\|f\|_2\|g\|_2\|h\|_\infty\,.$$

Combining (5.17) and (5.18), we complete the proof. $\qquad\square$

5.3. **Trilinear Forms.** Let f_1, f_2, f_3 be measurable functions supported on $1/16 \leq |\xi| \leq 39/16$. Define a trilinear form $\Lambda_{j,m,n}(f_1,f_2,f_3)$ by

$$(5.19) \qquad\qquad \Lambda_{j,m,n}(f_1,f_2,f_3) := \langle \mathcal{B}_{j,m,n}(\check{f}_1, \check{f}_2), \check{f}_3 \rangle\,.$$

By Lemma 5.1, Theorem 5.1 and Theorem 5.2 can be reduced to the following theorems respectively.

Theorem 5.3. *Let $d \geq 2$ and $\Lambda_{j,m,n}(f_1,f_2,f_3)$ be defined as in (5.19). If $|j| \leq m/(d-1)$, then there exists a constant C independent of j,m such that*

$$(5.20) \qquad |\Lambda_{j,m,n}(f_1,f_2,f_3)| \leq C2^{\frac{-(d-1)|j|-m}{2}} 2^{-\frac{m-(d-1)|j|}{6}}\|f_1\|_2\|f_2\|_2\|f_3\|_2\,,$$

holds for all $f_1, f_2, f_3 \in L^2$.

Theorem 5.4. *Let $d \geq 2$ and $\Lambda_{j,m,n}(f_1, f_2, f_3)$ be defined as in (5.19). If $|j| \geq m/(d-1)$, then there exist a positive number ε_0 and a constant C independent of j, m such that*

$$(5.21) \qquad |\Lambda_{j,m,n}(f_1, f_2, f_3)| \leq C \max \left\{ 2^{\frac{-(d-1)|j|+m}{2}}, 2^{-\varepsilon_0 m} \right\} \|f_1\|_2 \|f_2\|_2 \|\widehat{f_3}\|_\infty \,,$$

holds for all $f_1, f_2 \in L^2$ and $\widehat{f_3} \in L^\infty$ such that f_1, f_2, f_3 are supported on $1/16 \leq |\xi| \leq 39/16$.

A proof of Theorem 5.3 will be provided in Section 6 and a proof of Theorem 5.4 will be given in Section 9.

6. Stationary Phases and Trilinear Oscillatory Integrals

In Section 4, we see that Fourier series can help us to reduce the problem to the paraproduct case when $|m' - m| > 10^d$. This method does not work for the case when $|m - m'| \leq 10^d$. This is because the critical points of the phase function may happen in a neighborhood of $1/2 \leq |t| \leq 2$, say $1/4 \leq |t| \leq 5/2$, which provides a stationary phase for the Fourier integral $\mathfrak{m}_d$. This stationary phase gives a highly oscillatory factor in the integral. We expect a suitable decay from the highly oscillatory factor. In this section we should prove Theorem 5.3 by utilizing essentially a TT^* method.

Let $\Lambda_{j,m}(f_1, f_2, f_3) = \langle \mathcal{B}_{j,m}(\check{f_1}, \check{f_2}), \check{f_3} \rangle$. To prove Theorem 5.3, it suffices to prove the following L^2 estimate for the trilinear form $\Lambda_{j,m}(f_1, f_2, f_3)$,

$$(6.1) \qquad |\Lambda_{j,m}(f_1, f_2, f_3)| \leq C 2^{\frac{-(d-1)|j|-m}{2}} 2^{-\frac{m-(d-1)|j|}{6}} \|f_1\|_2 \|f_2\|_2 \|f_3\|_2 \,,$$

holds for all $f_1, f_2, f_3 \in L^2$. Clearly $\Lambda_{j,m}(f_1, f_2, f_3)$ can be expressed as if $j > 0$,

$$2^{-\frac{(d-1)j}{2}} \iint f_1(\xi)\widehat{\Phi}(\xi) f_2(\eta)\widehat{\Phi}(\eta) f_3\left(2^{-(d-1)j}\xi + \eta\right) \mathfrak{m}_d\left(2^m\xi, 2^m\eta\right) d\xi d\eta \,,$$

and if $j \leq 0$,

$$2^{\frac{(d-1)j}{2}} \iint f_1(\xi)\widehat{\Phi}(\xi) f_2(\eta)\widehat{\Phi}(\eta) f_3\left(\xi + 2^{(d-1)j}\eta\right) \mathfrak{m}_d\left(2^m\xi, 2^m\eta\right) d\xi d\eta \,,$$

Whenever $\xi, \eta \in \operatorname{supp}\widehat{\Phi}$, the second order derivative of the phase function $\phi_{m,\xi,\eta}(t) = 2^m(\xi t + \eta t^d)$ is comparable to 2^m. We only need to focus on the worst situation when there is a critical point of the phase function in a small neighborhood of $\operatorname{supp}\rho$. Thus the method of stationary phase yields

$$(6.2) \qquad \mathfrak{m}_d\left(2^m\xi, 2^m\eta\right) \sim 2^{-m/2} e^{ic_d 2^m \xi^{d/(d-1)}\eta^{-1/(d-1)}} \,,$$

where c_d is a constant depending only on d. Henceforth we reduce Theorem 5.3 to the following lemma.

Proposition 6.1. *Let $\Lambda_{j,m}^*$ be defined by if $j > 0$ then*
$$(6.3)$$
$$\Lambda_{j,m}^*(f_1, f_2, f_3) = \iint f_1(\xi)\widehat{\Phi}(\xi) f_2(\eta)\widehat{\Phi}(\eta) f_3\left(2^{-(d-1)j}\xi + \eta\right) e^{ic_d 2^m \xi^{d/(d-1)}\eta^{-1/(d-1)}} d\xi d\eta \,,$$

and if $j \leq 0$, then

(6.4)
$$\Lambda_{j,m}^*(f_1, f_2, f_3) = \iint f_1(\xi)\widehat{\Phi}(\xi) f_2(\eta)\widehat{\Phi}(\eta) f_3\left(\xi + 2^{(d-1)j}\eta\right) e^{ic_d 2^m \xi^{d/(d-1)}\eta^{-1/(d-1)}} d\xi d\eta \,,$$

Then there exists a positive constant C such that

(6.5)
$$\left|\Lambda_{j,m}^*(f_1, f_2, f_3)\right| \leq C 2^{-\frac{m-(d-1)|j|}{6}} \|f_1\|_2 \|f_2\|_2 \|f_3\|_2 \,,$$

holds for all $f_1, f_2, f_3 \in L^2$.

Proof. Without loss of generality, we assume that $\widehat{\Phi}$ is supported on $[1/2, 2]$ (or $[-2, -1/2]$). And we only give a proof for the case $j > 0$ since a similar argument yields the case $j \leq 0$. Let $\phi_{d,m}$ be a phase function defined by

$$\phi_{d,m}(\xi, \eta) = c_d \xi^{d/(d-1)} \eta^{-1/(d-1)} \,.$$

And let $b_1 = 1 - 2^{-(d-1)j}$ and $b_2 = 2^{-(d-1)j}$. Changing variable $\xi \mapsto \xi - \eta$ and $\eta \mapsto b_1\xi + b_2\eta$, we have that $\Lambda_{j,m}^*(f_1, f_2, f_3)$ equals

$$\iint f_1(\xi - \eta) f_2(b_1\xi + b_2\eta) f_3(\xi)\widehat{\Phi}(\xi - \eta)\widehat{\Phi}(b_1\xi + b_2\eta) e^{i2^m \phi_{d,m}(\xi-\eta, b_1\xi+b_2\eta)} d\xi d\eta.$$

Thus by Cauchy-Schwarz we dominate $|\Lambda_{j,m}^*(f_1, f_2, f_3)|$ by

$$\left\|\mathbf{T}_{d,j,m}(f_1, f_2)\right\|_2 \|f_3\|_2 \,,$$

where $\mathbf{T}_{d,j,m}$ is defined by

$$\mathbf{T}_{d,j,m}(f_1, f_2)(\xi) = \int f_1(\xi - \eta) f_2(b_1\xi + b_2\eta)\widehat{\Phi}(\xi - \eta)\widehat{\Phi}(b_1\xi + b_2\eta) e^{i2^m \phi_{d,m}(\xi-\eta, b_1\xi+b_2\eta)} d\eta.$$

It is easy to see that $\left\|\mathbf{T}_{d,j,m}(f_1, f_2)\right\|_2^2$ equals to

$$\int \left(\iint F(\xi, \eta_1, \eta_2) G(\xi, \eta_1, \eta_2) e^{i2^m \left(\phi_{d,m}(\xi-\eta_1, b_1\xi+b_2\eta_1) - \phi_{d,m}(\xi-\eta_2, b_1\xi+b_2\eta_2)\right)} d\eta_1 d\eta_2\right) d\xi \,,$$

where

$$F(\xi, \eta_1, \eta_2) = \left(f_1\widehat{\Phi}\right)(\xi - \eta_1)\overline{\left(f_1\widehat{\Phi}\right)(\xi - \eta_2)}$$
$$G(\xi, \eta_1, \eta_2) = \left(f_2\widehat{\Phi}\right)(b_1\xi + b_2\eta_1)\overline{\left(f_2\widehat{\Phi}\right)(b_1\xi + b_2\eta_2)} \,.$$

Changing variables $\eta_1 \mapsto \eta$ and $\eta_2 \mapsto \eta + \tau$, we see that $\left\|\mathbf{T}_{d,j,m}(f_1, f_2)\right\|_2^2$ equals to

$$\int \left(\iint F_\tau(\xi - \eta) G_\tau(b_1\xi + b_2\eta) e^{i2^m \left(\phi_{d,m}(\xi-\eta, b_1\xi+b_2\eta) - \phi_{d,m}(\xi-\eta-\tau, b_1\xi+b_2(\eta+\tau))\right)} d\xi d\eta\right) d\tau,$$

where

$$F_\tau(\cdot) = \left(f_1\widehat{\Phi}\right)(\cdot)\overline{\left(f_1\widehat{\Phi}\right)(\cdot - \tau)}$$
$$G_\tau(\cdot) = \left(f_2\widehat{\Phi}\right)(\cdot)\overline{\left(f_2\widehat{\Phi}\right)(\cdot + b_2\tau)} \,.$$

Changing coordinates to $(u, v) = (\xi - \eta, b_1\xi + b_2\eta)$, the inner integral becomes

(6.6)
$$\iint F_\tau(u) G_\tau(v) e^{i2^m \tilde{\mathbf{Q}}_\tau(u,v)} du dv \,,$$

where $\tilde{\mathbf{Q}}_\tau$ is defined by

$$\tilde{\mathbf{Q}}_\tau(u, v) = \phi_{d,m}(u, v) - \phi_{d,m}(u - \tau, v + b_2\tau).$$

When j is large enough, the mean value theorem yields

(6.7) $$\left|\partial_u \partial_v \tilde{\mathbf{Q}}_\tau(u, v)\right| \geq C\tau,$$

if $u, v, u - \tau, u + b_2\tau \in \operatorname{supp}\widehat{\Phi}$.

A well-known Hörmander theorem on the non-degenerate phase [13, 23] gives that (6.6) is estimated by

$$C \min\left\{1, 2^{-m/2}|\tau|^{-1/2}\right\}\left\|F_\tau\right\|_2 \left\|G_\tau\right\|_2.$$

Hence by Cauchy-Schwarz inequality $\left\|\mathbf{T}_{d,j,m}(f_1, f_2)\right\|_2^2$ is bounded by

$$\tau_0 \|f_1\|_2^2 \|f_2\|_2^2 + C \int_{\tau_0 < |\tau| < 10} \min\left\{1, 2^{-m/2}|\tau|^{-1/2}\right\}\left\|F_\tau\right\|_2 \left\|G_\tau\right\|_2 d\tau$$

for any $\tau_0 > 0$. By one more use of Cauchy-Schwarz inequality, $\left\|\mathbf{T}_{d,j,m}(f_1, f_2)\right\|_2^2$ is dominated by

$$\left(\tau_0 + C\tau_0^{-1/2} 2^{-m/2} 2^{(d-1)j/2}\right) \|f_1\|_2^2 \|f_2\|_2^2,$$

for any $\tau_0 > 0$. Thus we have

(6.8) $$\left|\Lambda_{j,m}^*(f_1, f_2, f_3)\right| \leq C 2^{\frac{(d-1)j-m}{6}} \|f_1\|_2 \|f_2\|_2 \|f_3\|_2.$$

This completes the proof of Proposition 6.1.

$$\square$$

It is easy to see that

(6.9) $$\left|\Lambda_{j,m}^*(f_1, f_2, f_3)\right| \leq C 2^{-\varepsilon m} \|f_1\|_2 \|f_2\|_2 \|f_3\|_2$$

fails for all $|j| \geq m/(d-1)$. Indeed, let us only consider the case $j > m/(d-1)$. Assume that (6.9) holds for all $j > m/(d-1)$. Let $j \to \infty$, then (6.9) implies

(6.10) $$\left|\Lambda_m^*(f_1, f_2, f_3)\right| \leq C 2^{-\varepsilon m} \|f_1\|_2 \|f_2\|_2 \|f_3\|_2,$$

where

$$\Lambda_m^*(f_1, f_2, f_3) = \iint f_1(\xi)\widehat{\Phi}(\xi) f_2(\eta)\widehat{\Phi}(\eta) f_3(\eta) e^{ic_d 2^m \xi^{d/(d-1)} \eta^{-1/(d-1)}} d\xi d\eta.$$

Simply taking $f_2 = f_3$, we obtain

(6.11) $$\sup_{\eta \sim 1} \left|\int f_1(\xi)\widehat{\Phi}(\xi) e^{ic_d 2^m \xi^{d/(d-1)} \eta^{-1/(d-1)}} d\xi\right| \leq C 2^{-\varepsilon m} \|f_1\|_2.$$

This clearly can not be true and hence we get a contradiction. Therefore, (6.9) does not hold for all $j > m/(d-1)$. From this fact, we know that the TT^* method can not work for the case $|j| > m/(d-1)$. In the following sections, we have to introduce a concept of uniformity and employ a "quadratic" Fourier analysis.

7. Uniformity

We introduce a concept related to a notion of uniformity employed by Gowers [10]. A similar uniformity was utilized in [4]. Let $\sigma \in (0, 1]$, let $\mathcal{Q}$ be a collection of some real-valued measurable functions, and fix a bounded interval $\mathbf{I}$ in $\mathbb{R}$.

Definition 7.1. *A function $f \in L^2(\mathbf{I})$ is σ-uniform in $\mathcal{Q}$ if*

$$(7.1) \qquad \left| \int_{\mathbf{I}} f(\xi) e^{-iq(\xi)} d\xi \right| \leq \sigma \|f\|_{L^2(\mathbf{I})}$$

for all $q \in \mathcal{Q}$. Otherwise, f is said to be σ-nonuniform in $\mathcal{Q}$.

Theorem 7.1. *Let L be a bounded sub-linear functional from $L^2(\mathbf{I})$ to $\mathbb{C}$, let $\mathbf{S}_\sigma$ be the set of all functions that are σ-uniform in $\mathcal{Q}$, and let*

$$(7.2) \qquad U_\sigma = \sup_{f \in \mathbf{S}_\sigma} \frac{|L(f)|}{\|f\|_{L^2(\mathbf{I})}} \, .$$

Then for all functions in $L^2(\mathbf{I})$,

$$(7.3) \qquad |L(f)| \leq \max \left\{ U_\sigma, 2\sigma^{-1} Q \right\} \|f\|_{L^2(\mathbf{I})} \, ,$$

where

$$(7.4) \qquad Q = \sup_{q \in \mathcal{Q}} \left| L(e^{iq}) \right| .$$

Proof. Clearly the complement $\mathbf{S}_\sigma^c$ is a set of all functions that are σ-nonuniform in $\mathcal{Q}$. Let us set

$$A := \sup_{f \in L^2(\mathbf{I})} \frac{|L(f)|}{\|f\|_{L^2(\mathbf{I})}} \quad \text{and} \quad A_1 := \sup_{f \in \mathbf{S}_\sigma^c} \frac{|L(f)|}{\|f\|_{L^2(\mathbf{I})}} \, .$$

Clearly $A = \max\{A_1, U_\sigma\}$. In order to obtain (7.3), it suffices to prove that if $U_\sigma < A_1$, then

$$(7.5) \qquad A_1 \leq 2\sigma^{-1} Q \, .$$

For any $\varepsilon > 0$, there exists a function $f \in \mathbf{S}_\sigma^c$ such that

$$(7.6) \qquad (A_1 - \varepsilon) \|f\|_{L^2(\mathbf{I})} \leq |L(f)| \, .$$

Let $\langle \cdot, \cdot \rangle_{\mathbf{I}}$ be an inner product on $L^2(\mathbf{I})$ defined by

$$\langle f, g \rangle_{\mathbf{I}} = \int_{\mathbf{I}} f(x) \overline{g(x)} dx \, ,$$

for all $f, g \in L^2(\mathbf{I})$. Since f is σ-nonuniform in $\mathcal{Q}$, there exists a function q in $\mathcal{Q}$ such that

$$(7.7) \qquad \left| \langle f, e^{iq} \rangle_{\mathbf{I}} \right| \geq \sigma \|f\|_{L^2(\mathbf{I})} \, .$$

Let $g \in L^2(\mathbf{I})$ such that $g \perp e^{iq}$ and $\|g\|_{L^2(\mathbf{I})} = 1$. Then we can write f as

$$(7.8) \qquad f = \langle f, g \rangle_{\mathbf{I}} \, g + \frac{\langle f, e^{iq} \rangle_{\mathbf{I}}}{|\mathbf{I}|} e^{iq} \, .$$

Sub-linearity of L and the triangle inequality then yield

(7.9) $$\left|L(f)\right| \leq \left|\langle f, g\rangle_{\mathbf{I}}\right|\left|L(g)\right| + |\mathbf{I}|^{-1}\left|\langle f, e^{iq}\rangle_{\mathbf{I}}\right|\left|L(e^{iq})\right|.$$

Notice that $A = A_1$ if $U_\sigma < A_1$ and

(7.10) $$\langle f, f\rangle_{\mathbf{I}} = \left|\langle f, g\rangle_{\mathbf{I}}\right|^2 + |\mathbf{I}|^{-1}\left|\langle f, e^{iq}\rangle_{\mathbf{I}}\right|^2.$$

Then from (7.6) and (7.9), we have

(7.11) $$(A_1 - \varepsilon)\|f\|_{L^2(\mathbf{I})} \leq A_1\|f\|_{L^2(\mathbf{I})}\sqrt{1 - \frac{\left|\langle f, e^{iq}\rangle_{\mathbf{I}}\right|^2}{|\mathbf{I}|\langle f, f\rangle_{\mathbf{I}}}} + |\mathbf{I}|^{-1}\left|\langle f, e^{iq}\rangle_{\mathbf{I}}\right|Q.$$

Applying the elementary inequality $\sqrt{1-x} \leq 1 - x/2$ if $0 \leq x \leq 1$, we then get

(7.12) $$A_1 \leq \frac{2\|f\|_{L^2(\mathbf{I})}}{\left|\langle f, e^{iq}\rangle_{\mathbf{I}}\right|}Q + \varepsilon|I|\frac{2\|f\|_{L^2(\mathbf{I})}^2}{\left|\langle f, e^{iq}\rangle_{\mathbf{I}}\right|^2}.$$

From (7.7), we have

(7.13) $$A_1 \leq 2\sigma^{-1}Q + 2\varepsilon|I|\sigma^{-2}.$$

Now let $\varepsilon \to 0$ and we then obtain (7.5). Therefore we complete the proof. $\qquad\square$

8. Estimates of the trilinear forms, Case $j > 0$

Without loss of generality, in the following sections we assume that f_i is supported on $\mathbf{I}_i$ for $i \in \{1, 2, 3\}$, where $\mathbf{I}_i$ is either $[1/16, 39/16]$ or $[-39/16, -1/16]$. Let $\mathcal{Q}_1$ be a set of some functions defined by

(8.1) $$\mathcal{Q}_1 = \left\{a\xi^{d/d-1} + b\xi : 2^{m-100} \leq |a| \leq 2^{m+100} \text{ and } a, b \in \mathbb{R}\right\}.$$

Proposition 8.1. *Let f_1 be σ-uniform in $\mathcal{Q}_1$. And let $j > 0$ and $\Lambda_{j,m,n}(f_1, f_2, f_3)$ be defined as in (5.19). Then there exists a constant C independent of j, m, n, f_1 such that*

(8.2) $$|\Lambda_{j,m,n}(f_1, f_2, f_3)| \leq C2^{-\frac{(d-1)j}{2} - \frac{m}{2}}\max\left\{2^{-100m}, 2^{\frac{-(d-1)j+m}{2}}, \sigma\right\}\prod_{i=1}^{3}\|f_i\|_{L^2(\mathbf{I}_i)},$$

holds for all $f_2 \in L^2(\mathbf{I}_2)$ and $f_3 \in L^2(\mathbf{I}_3)$.

Proof. Let $\mathbf{1}_{m,l} = \mathbf{1}_{\mathbf{I}_{m,l}}$ and let $\mathcal{B}_{j,m,n,\ell}$ be a bilinear operator defined by

$$\mathcal{B}_{j,m,n,\ell}(f, g)(x) = \mathcal{B}_{j,m,n}(f, g)(x)\mathbf{1}_{m,\ell}(x),$$

for all f, g. Decompose $\Lambda_{j,m,n}(f_1, f_2, f_3)$ into $\sum_\ell \Lambda_{j,m,n,\ell}$, where

$$\Lambda_{j,m,n,\ell}(f_1, f_2, f_3) = \left\langle \mathcal{B}_{j,m,n,\ell}(\check{f}_1, \check{f}_2), \check{f}_3\right\rangle.$$

Let $\alpha_{m,\ell}$ be a fixed point in the interval $\mathbf{I}_{m,\ell}$. And set $F_{\Phi_1,j,m,\ell}(x, t)$ to be

$$F_{\Phi_1,j,m,\ell}(x, t) := R_{\Phi_1}\check{f}_1(2^{-(d-1)j}x - 2^m t) - R_{\Phi_1}\check{f}_1(2^{-(d-1)j}\alpha_{m,\ell} - 2^m t)$$

Split $\mathcal{B}_{j,m,n,\ell}(\check{f}_1, \check{f}_2)$ into two terms:

$$\mathcal{B}_{j,m,n,\ell}^{(1)}\left(\check{f}_1, \check{f}_2\right) + \mathcal{B}_{j,m,n,\ell}^{(2)}\left(\check{f}_1, \check{f}_2\right),$$

where $\mathcal{B}^{(1)}_{j,m,n,\ell}\left(\check{f}_1, \check{f}_2\right)$ is equal to

$$2^{-(d-1)j/2}\int_{\mathbb{R}} F_{\Phi_1,j,m,\ell}(x,t)R_{\Phi_1}\check{f}_2(x-2^m t^d)\rho(t)dt\left(\mathbf{1}^*_{(d-1)j+m,n}(x)\mathbf{1}_{m,\ell}(x)\right)$$

and $\mathcal{B}^{(2)}_{j,m,n,\ell}\left(\check{f}_1, \check{f}_2\right)$ equals to

$$2^{-(d-1)j/2}\int_{\mathbb{R}} R_{\Phi_1}\check{f}_1(2^{-(d-1)j}\alpha_{m,\ell}-2^m t)R_{\Phi_1}\check{f}_2(x-2^m t^d)\rho(t)dt\left(\mathbf{1}^*_{(d-1)j+m,n}(x)\mathbf{1}_{m,\ell}(x)\right)\ .$$

For $i = 1, 2$, let $\Lambda^{(i)}_{j,m,n}(f_1, f_2, f_3)$ denote

$$\sum_\ell \left\langle \mathcal{B}^{(i)}_{j,m,n,\ell}\left(\check{f}_1, \check{f}_2\right), \check{f}_3\right\rangle\ .$$

We now start to prove that

$$(8.3) \qquad \left|\Lambda^{(1)}_{j,m,n}(f_1, f_2, f_3)\right| \leq 2^{-\frac{(d-1)j}{2}}2^{-(d-1)j+m}\left\|\check{f}_1\right\|_\infty\left\|\check{f}_2\right\|_2\left\|\check{f}_3\right\|_2\ .$$

The mean value theorem and the smoothness of Φ_1 yield that for $x \in \mathbf{I}_{m,\ell}$,

$$(8.4) \qquad |F_{\Phi_1,j,m,\ell}(x,t)| \leq C\left\|\check{f}_1\right\|_\infty 2^{-(d-1)j}|x-\alpha_{m,\ell}| \leq C2^{-(d-1)j+m}\left\|\check{f}_1\right\|_\infty\ .$$

Because $|t| \sim 1$ when $t \in \operatorname{supp}\rho$, $\mathcal{B}^{(1)}_{j,m,n,\ell}\left(\check{f}_1, \check{f}_2\right)$ can be written as
$$(8.5)$$
$$2^{-\frac{(d-1)j}{2}}\int_{\mathbb{R}} F_{\Phi_1,j,m,\ell}(x,t)\sum_{\ell_0}\left(\mathbf{1}_{m,\ell+\ell_0}R_{\Phi_1}\check{f}_2\right)(x-2^m t^d)\rho(t)dt\left(\mathbf{1}^*_{(d-1)j+m,n}(x)\mathbf{1}_{m,\ell}(x)\right)\ ,$$

where ℓ_0 is an integer between -10 and 10. Putting absolute value throughout and applying (8.4) plus Cauchy-Schwarz inequality, we then estimate $\left|\Lambda^{(1)}_{j,m,n}(f_1, f_2, f_3)\right|$ by

$$C2^{-\frac{(d-1)j}{2}}2^{-(d-1)j+m}\left\|\check{f}_1\right\|_\infty\sum_{\ell_0=-10}^{10}\sum_\ell\left\|\mathbf{1}_{m,\ell+\ell_0}R_{\Phi_1}\check{f}_2\right\|_2\left\|\mathbf{1}_{m,\ell}\check{f}_3\right\|_2\ ,$$

which clearly gives (8.3) by one more use of Cauchy-Schwarz inequality.

We now prove that

$$(8.6) \qquad \left|\Lambda^{(1)}_{j,m,n}(f_1, f_2, f_3)\right| \leq 2^{-\frac{(d-1)j}{2}}2^{-m}\left\|\check{f}_1\right\|_1\left\|\check{f}_2\right\|_2\left\|\check{f}_3\right\|_2\ .$$

From (8.5), we get that $\Lambda^{(1)}_{j,m,n}(f_1, f_2, f_3)$ equals to

$$2^{-\frac{(d-1)j}{2}}\sum_{\ell_0=-10}^{10}\sum_\ell \Lambda_{j,m,n,\ell_0,\ell,1}(f_1, f_2, f_3) - \Lambda_{j,m,n,\ell_0,\ell,2}(f_1, f_2, f_3)\,,$$

where $\Lambda_{j,m,n,\ell_0,\ell,1}(f_1, f_2, f_3)$ is equal to

$$\int_{\mathbb{R}^2} R_{\Phi_1}\check{f}_1(2^{-(d-1)j}x-2^m t)\left(\mathbf{1}_{m,\ell+\ell_0}R_{\Phi_1}\check{f}_2\right)(x-2^m t^d)\rho(t)\left(\mathbf{1}^*_{(d-1)j+m,n}\mathbf{1}_{m,\ell}\check{f}_3\right)(x)dtdx$$

and $\Lambda_{j,m,n,\ell_0,\ell,2}(f_1, f_2, f_3)$ equals to

$$\int_{\mathbb{R}^2} R_{\Phi_1}\check{f}_1(2^{-(d-1)j}\alpha_{m,\ell} - 2^m t)\big(\mathbf{1}_{m,\ell+\ell_0}R_{\Phi_1}\check{f}_2\big)(x - 2^m t^d)\rho(t)\big(\mathbf{1}^*_{(d-1)j+m,n}\mathbf{1}_{m,\ell}\check{f}_3\big)(x)dtdx.$$

Cauchy-Schwarz inequality yields that

$$(8.7) \qquad |\Lambda_{j,m,n,\ell_0,\ell,2}(f_1, f_2, f_3)| \leq C2^{-m} \left\|\check{f}_1\right\|_1 \left\|\mathbf{1}_{m,\ell+\ell_0}R_{\Phi_1}\check{f}_2\right\|_2 \left\|\mathbf{1}_{m,\ell}\check{f}_3\right\|_2.$$

In order to obtain a similar estimate for $\Lambda_{j,m,n,\ell_0,\ell,1}(f_1, f_2, f_3)$, we change variables by $u = 2^{-(d-1)j}x - 2^m t$ and $v = x - 2^m t^d$ to express $\Lambda_{j,m,n,\ell_0,\ell,1}(f_1, f_2, f_3)$ as

$$\iint R_{\Phi_1}\check{f}_1(u) \big(\mathbf{1}_{m,\ell+\ell_0}R_{\Phi_1}\check{f}_2\big)(v)\rho(t(u,v)) \big(\mathbf{1}^*_{(d-1)j+m,n}\mathbf{1}_{m,\ell}\check{f}_3\big)(x(u,v))\frac{dudv}{J(u,v)},$$

where $J(u,v)$ is the Jocobian $\frac{\partial(u,v)}{\partial(x,t)}$. It is easy to see that the Jocobian $\frac{\partial(u,v)}{\partial(x,t)} \sim 2^m$. As we did for $\Lambda_{j,m,n,\ell_0,\ell,1}$, we dominate the previous integral by

$$C2^{-m} \int \left|R_{\Phi_1}\check{f}_1(u)\right| \left\|\mathbf{1}_{m,\ell+\ell_0}R_{\Phi_1}\check{f}_2\right\|_2 \left(\int \left|(\mathbf{1}_{m,\ell}\check{f}_3)(x(u,v))\rho(t(u,v))\right|^2 dv\right)^{\frac{1}{2}} du.$$

Notice that $|\partial x/\partial v| \sim 1$ whenever $t \in \mathrm{supp}\rho$. We then estimate

$$(8.8) \qquad |\Lambda_{j,m,n,\ell_0,\ell,1}(f_1, f_2, f_3)| \leq C2^{-m} \left\|\check{f}_1\right\|_1 \left\|\mathbf{1}_{m,\ell+\ell_0}R_{\Phi_1}\check{f}_2\right\|_2 \left\|\mathbf{1}_{m,\ell}\check{f}_3\right\|_2,$$

(8.6) follows from (8.7) and (8.8). An interpolation of (8.3) and (8.6) then yields

$$(8.9) \qquad \left|\Lambda^{(1)}_{j,m,n}(f_1, f_2, f_3)\right| \leq C2^{-\frac{(d-1)j}{2}-\frac{m}{2}}2^{\frac{-(d-1)j+m}{2}} \prod_{i=1}^3 \|f_i\|_{L^2(\mathbf{I}_i)}.$$

We now turn to prove that

$$(8.10) \qquad \left|\Lambda^{(2)}_{j,m,n}(f_1, f_2, f_3)\right| \leq C_N 2^{-\frac{(d-1)j}{2}-\frac{m}{2}} \max\left\{2^{-100m}, \sigma\right\} \prod_{i=1}^3 \|f_i\|_{L^2(\mathbf{I}_i)}.$$

In dual frequency variables, $\Lambda^{(2)}_{j,m,n}(f_1, f_2, f_3)$ can be expressed as

$$\sum_{\ell_0=-10}^{10} \sum_{\ell} 2^{-\frac{(d-1)j}{2}} \iint f_1(\xi)\widehat{\Phi_1}(\xi)e^{2\pi i 2^{-(d-1)j}\alpha_{m,\ell}\xi} \widehat{F_{2,m,\ell_0,\ell}}(\eta)\mathfrak{m}(\xi,\eta)\widehat{F_{3,m,n,\ell}}(\eta)d\xi d\eta,$$

where

$$(8.11) \qquad \mathfrak{m}(\xi,\eta) = \int \rho(t)e^{-2\pi i(2^m\xi t + 2^m\eta t^d)}dt$$

$$F_{2,m,\ell_0,\ell} = \mathbf{1}_{m,\ell+\ell_0}R_{\Phi_1}\check{f}_2 \quad \text{and} \quad F_{3,m,n,\ell} = \mathbf{1}^*_{(d-1)j+m,n}\mathbf{1}_{m,\ell}\check{f}_3.$$

If η is not in a small neighborhood of $\widehat{\Phi_1}$, then there is no critical point of the phase function $\phi_{\xi,\eta}(t) = \xi t + \eta t^d$ occurring in a small neighborhood of $\mathrm{supp}\,\rho$. Integration by parts gives a rapid decay $O(2^{-Nm})$ for $\mathfrak{m}$. Thus in this case, we dominate

$\left| \Lambda_{j,m,n}^{(2)}(f_1, f_2, f_3) \right|$ by

(8.12)
$$C_N 2^{-Nm} \prod_{i=1}^{3} \|f_i\|_{L^2(\mathbf{I}_i)} \,,$$

for any positive integer N. We now only need to consider the worst case when there is a critical point of the phse function $\phi_{\xi,\eta}(t) = \xi t + \eta t^d$ in a small neighborhood of supp ρ. In this case, η must be in a small neighborhood of $\widehat{\Phi_1}$ and the stationary phase method gives

(8.13)
$$\mathfrak{m}(\xi, \eta) \sim 2^{-m/2} e^{2\pi i c_d 2^m \eta^{-\frac{1}{d-1}} \xi^{d/(d-1)}} \,,$$

where c_d is a constant depending on d only. Thus the principle term of $\Lambda_{j,m,n}^{(2)}(f_1, f_2, f_3)$ is

$$\sum_{\ell_0=-10}^{10} \sum_{\ell} 2^{-\frac{(d-1)j}{2} - \frac{m}{2}} \iint f_1(\xi) \widehat{\Phi_1}(\xi) e^{i\phi_{d,m,\eta}(\xi)} \overline{\widehat{F_{2,m,\ell_0,\ell}}}(\eta) \widehat{\Phi_2}(\eta) \overline{\widehat{F_{3,m,n,\ell}}}(\eta) d\xi d\eta \,,$$

where $\widehat{\Phi_2}$ is a Schwartz function supported on a small neighborhood of $\widehat{\Phi_1}$, and

$$\phi_{d,m,\eta}(\xi) = 2\pi c_d 2^m \eta^{-\frac{1}{d-1}} \xi^{d/(d-1)} + 2\pi 2^{-(d-1)j} \alpha_{m,\ell} \xi \,.$$

The key point is that the integral in the previous expression can be viewed as an inner product of $F_{3,m,n,\ell}$ and $\mathcal{M}F_{2,m,\ell_0,\ell}$, where $\mathcal{M}$ is a multiplier operator defined by

$$\widehat{\mathcal{M}f}(\eta) = \mathfrak{m}_{d,j,m}(\eta) \hat{f}(\eta) \,.$$

Here the multiplier $\mathfrak{m}_{d,j,m}$ is given by

(8.14)
$$\mathfrak{m}_{d,j,m}(\eta) = \int f_1(\xi) \widehat{\Phi_1}(\xi) e^{i\phi_{d,m,\eta}(\xi)} d\xi \,.$$

Observe that $\phi_{d,m,\eta}(\xi) + b\xi$ is in $\mathcal{Q}_1$ for any $b \in \mathbb{R}$ and $\eta \in \text{supp} \, \widehat{\Phi_2}$. Thus σ-uniformity in $\mathcal{Q}_1$ of f_1 yields

(8.15)
$$\|\mathfrak{m}_{d,j,m}\|_\infty \leq C\sigma \|f_1\|_{L^2(\mathbf{I}_1)} \,.$$

And henceforth we dominate $\Lambda_{j,m,n}^{(2)}(f_1, f_2, f_3)$ by

$$\sum_{\ell_0=-10}^{10} \sum_{\ell} 2^{-\frac{(d-1)j}{2} - \frac{m}{2}} \sigma \|f_1\|_{L^2(\mathbf{I}_1)} \|F_{2,m,\ell_0,\ell}\|_2 \|F_{3,m,n,\ell}\|_2 \,,$$

which clearly is bounded by

(8.16)
$$2^{-\frac{(d-1)j}{2} - \frac{m}{2}} \sigma \prod_{i=1}^{3} \|f_i\|_{L^2(\mathbf{I}_i)} \,.$$

Now (8.10) follows from (8.12) and (8.16). Combining (8.9) and (8.10), we finish the proof. $\qquad \square$

Corollary 8.1. *Let* $\Lambda_{j,m,n}(f_1, f_2, f_3)$ *be defined as in (5.19). Then there exists a constant C independent of j, m, n such that*

$$(8.17) \quad |\Lambda_{j,m,n}(f_1, f_2, f_3)| \leq C \max \left\{ 2^{-100m}, 2^{\frac{-(d-1)j+m}{2}}, \sigma \right\} \|f_1\|_{L^2(\mathbf{I}_1)} \|f_2\|_{L^2(\mathbf{I}_1)} \|\widehat{f_3}\|_\infty,$$

holds for all $f_1 \in L^2(\mathbf{I}_1)$ which are σ-uniform in $\mathcal{Q}_1$, $f_2 \in L^2(\mathbf{I}_2)$ and $\widehat{f_3} \in L^\infty$.

Proof. Since there is a smooth restriction factor $\mathbf{1}^*_{(d-1)j+m,n}$ in the definition of $\mathcal{B}_{j,m,m}$, the right hand side of (8.2) can be sharpen to

$$(8.18) \quad C 2^{-\frac{(d-1)j}{2}-\frac{m}{2}} \max \left\{ 2^{-100m}, 2^{\frac{-(d-1)j+m}{2}}, \sigma \right\} \|f_1\|_{L^2(\mathbf{I}_1)} \|f_2\|_{L^2(\mathbf{I}_2)} \left\| \mathbf{1}^{**}_{(d-1)j+m,n} \check{f_3} \right\|_2,$$

which is clearly bounded by

$$C \max \left\{ 2^{-100m}, 2^{\frac{-(d-1)j+m}{2}}, \sigma \right\} \|f_1\|_{L^2(\mathbf{I}_1)} \|f_2\|_{L^2(\mathbf{I}_1)} \|\widehat{f_3}\|_\infty.$$

$\square$

Proposition 8.2. *Let* $\Lambda_{j,m,n}(f_1, f_2, f_3)$ *be defined as in (5.19). Then there exists a constant C independent of j, m, n such that*

$$(8.19) \qquad \left| \Lambda_{j,m,n}(e^{iq_1}, f_2, f_3) \right| \leq C 2^{-\frac{\mathfrak{D}(d-1)m}{2}} \|f_2\|_{L^2(\mathbf{I}_2)} \|\widehat{f_3}\|_\infty,$$

holds for all $q_1 \in \mathcal{Q}_1$, $f_2 \in L^2(\mathbf{I}_2)$ and $\widehat{f_3} \in L^\infty$, where $\mathfrak{D}(d-1)$ is a positive constant defined in (10.4).

A proof of Proposition 8.2 will be provided in Section 10.

9. ESTIMATES OF THE TRILINEAR FORMS, CASE $j < 0$

Let $\mathcal{Q}_2$ be a set of some functions defined by

$$(9.1) \qquad \mathcal{Q}_2 = \left\{ a\eta^{-\frac{1}{d-1}} + b\eta : 2^{m-100} \leq |a| \leq 2^{m+100} \text{ and } a, b \in \mathbb{R} \right\}.$$

Proposition 9.1. *Let f_2 be σ-uniform in $\mathcal{Q}_2$. And let $j \leq 0$ and $\Lambda_{j,m,n}(f_1, f_2, f_3)$ be defined as in (5.19). Then there exists a constant C independent of j, m, n, f_1 such that*

$$(9.2) \qquad |\Lambda_{j,m,n}(f_1, f_2, f_3)| \leq C 2^{\frac{(d-1)j}{2}-\frac{m}{2}} \max \left\{ 2^{-100m}, 2^{\frac{(d-1)j+m}{2}}, \sigma \right\} \prod_{i=1}^{3} \|f_i\|_{L^2(\mathbf{I}_i)},$$

holds for all $f_1 \in L^2(\mathbf{I}_2)$ and $f_3 \in L^2(\mathbf{I}_3)$.

Proof. Let $\mathbf{1}_{m,l} = \mathbf{1}_{\mathbf{I}_{m,l}}$ and let $\mathcal{B}_{j,m,n,\ell}$ be a bilinear operator defined by

$$\mathcal{B}_{j,m,n,\ell}(f, g)(x) = \mathcal{B}_{j,m,n}(f, g)(x) \mathbf{1}_{m,\ell}(x),$$

for all f, g. Decompose $\Lambda_{j,m,n}(f_1, f_2, f_3)$ into $\sum_\ell \Lambda_{j,m,n,\ell}$, where

$$\Lambda_{j,m,n,\ell}(f_1, f_2, f_3) = \left\langle \mathcal{B}_{j,m,n,\ell}(\check{f_1}, \check{f_2}), \check{f_3} \right\rangle.$$

Let $\alpha_{m,\ell}$ be a fixed point in the interval $\mathbf{I}_{m,\ell}$. And set $G_{\Phi_1, j, m, \ell}(x, t)$ to be

$$G_{\Phi_1, j, m, \ell}(x, t) := R_{\Phi_1} \check{f_2}(2^{(d-1)j} x - 2^m t^d) - R_{\Phi_1} \check{f_2}(2^{(d-1)j} \alpha_{m,\ell} - 2^m t^d)$$

Split $\mathcal{B}_{j,m,n,\ell}(\check{f}_1, \check{f}_2)$ into two terms:

$$\mathcal{B}^{(1)}_{j,m,n,\ell}\left(\check{f}_1, \check{f}_2\right) + \mathcal{B}^{(2)}_{j,m,n,\ell}\left(\check{f}_1, \check{f}_2\right),$$

where $\mathcal{B}^{(1)}_{j,m,n,\ell}\left(\check{f}_1, \check{f}_2\right)$ is equal to

$$2^{(d-1)j/2}\int_{\mathbb{R}} R_{\Phi_1}\check{f}_1\left(x - 2^m t\right) G_{\Phi_1,j,m,\ell}(x,t)\rho(t)dt \left(\mathbf{1}^*_{(d-1)|j|+m,n}(x)\mathbf{1}_{m,\ell}(x)\right)$$

and $\mathcal{B}^{(2)}_{j,m,n,\ell}\left(\check{f}_1, \check{f}_2\right)$ equals to

$$2^{-(d-1)j/2}\int_{\mathbb{R}} R_{\Phi_1}\check{f}_1\left(x - 2^m t\right) R_{\Phi_1}\check{f}_2(2^{(d-1)j}\alpha_{m,\ell} - 2^m t^d)\rho(t)dt \left(\mathbf{1}^*_{(d-1)|j|+m,n}(x)\mathbf{1}_{m,\ell}(x)\right).$$

For $i = 1, 2$, let $\Lambda^{(i)}_{j,m,n}(f_1, f_2, f_3)$ denote

$$\sum_\ell \left\langle \mathcal{B}^{(i)}_{j,m,n,\ell}\left(\check{f}_1, \check{f}_2\right), \check{f}_3 \right\rangle.$$

We now start to prove that

(9.3) $$\left|\Lambda^{(1)}_{j,m,n}(f_1, f_2, f_3)\right| \leq 2^{\frac{(d-1)j}{2}} 2^{(d-1)j+m} \left\|\check{f}_1\right\|_2 \left\|\check{f}_2\right\|_\infty \left\|\check{f}_3\right\|_2.$$

The mean value theorem and the smoothness of Φ_1 yield that for $x \in \mathbf{I}_{m,\ell}$,

(9.4) $$|G_{\Phi_1,j,m,\ell}(x,t)| \leq C\left\|\check{f}_2\right\|_\infty 2^{(d-1)j}|x - \alpha_{m,\ell}| \leq C2^{(d-1)j+m}\left\|\check{f}_2\right\|_\infty.$$

Because $|t| \sim 1$ when $t \in \operatorname{supp}\rho$, $\mathcal{B}^{(1)}_{j,m,n,\ell}\left(\check{f}_1, \check{f}_2\right)$ can be written as
(9.5)
$$2^{\frac{(d-1)j}{2}}\int_{\mathbb{R}} G_{\Phi_1,j,m,\ell}(x,t)\sum_{\ell_0}\left(\mathbf{1}_{m,\ell+\ell_0} R_{\Phi_1}\check{f}_1\right)(x - 2^m t)\rho(t)dt \left(\mathbf{1}^*_{(d-1)|j|+m,n}(x)\mathbf{1}_{m,\ell}(x)\right),$$

where ℓ_0 is an integer between -10 and 10. Putting absolute value throughout and applying (9.4) plus Cauchy-Schwarz inequality, we then estimate $\left|\Lambda^{(1)}_{j,m,n}(f_1, f_2, f_3)\right|$ by

$$C2^{\frac{(d-1)j}{2}} 2^{(d-1)j+m} \left\|\check{f}_2\right\|_\infty \sum_{\ell_0=-10}^{10}\sum_\ell \left\|\mathbf{1}_{m,\ell+\ell_0} R_{\Phi_1}\check{f}_1\right\|_2 \left\|\mathbf{1}_{m,\ell}\check{f}_3\right\|_2,$$

which clearly gives (9.3) by one more use of Cauchy-Schwarz inequality.

We now prove that

(9.6) $$\left|\Lambda^{(1)}_{j,m,n}(f_1, f_2, f_3)\right| \leq 2^{\frac{(d-1)j}{2}} 2^{-m} \left\|\check{f}_1\right\|_2 \left\|\check{f}_2\right\|_1 \left\|\check{f}_3\right\|_2.$$

From (9.5), we get that $\Lambda^{(1)}_{j,m,n}(f_1, f_2, f_3)$ equals to

$$2^{\frac{(d-1)j}{2}}\sum_{\ell_0=-10}^{10}\sum_\ell \Lambda_{j,m,n,\ell_0,\ell,1}(f_1, f_2, f_3) - \Lambda_{j,m,n,\ell_0,\ell,2}(f_1, f_2, f_3),$$

where $\Lambda_{j,m,n,\ell_0,\ell,1}(f_1, f_2, f_3)$ is equal to

$$\int_{\mathbb{R}^2} \left(\mathbf{1}_{m,\ell+\ell_0} R_{\Phi_1} \check{f}_1\right)(x - 2^m t) R_{\Phi_1} \check{f}_2 \left(2^{(d-1)j}x - 2^m t^d\right) \rho(t) \left(\mathbf{1}^*_{(d-1)|j|+m,n} \mathbf{1}_{m,\ell} \check{f}_3\right)(x) dt dx$$

and $\Lambda_{j,m,n,\ell_0,\ell,2}(f_1, f_2, f_3)$ equals to

$$\int_{\mathbb{R}^2} \left(\mathbf{1}_{m,\ell+\ell_0} R_{\Phi_1} \check{f}_1\right)(x - 2^m t) R_{\Phi_1} \check{f}_2 \left(2^{(d-1)j}\alpha_{m,\ell} - 2^m t^d\right) \rho(t) \left(\mathbf{1}^*_{(d-1)|j|+m,n} \mathbf{1}_{m,\ell} \check{f}_3\right)(x) dt dx.$$

Cauchy-Schwarz inequality yields that

$$(9.7) \qquad |\Lambda_{j,m,n,\ell_0,\ell,2}(f_1, f_2, f_3)| \leq C 2^{-m} \left\|\mathbf{1}_{m,\ell+\ell_0} R_{\Phi_1} \check{f}_1\right\|_2 \left\|\check{f}_2\right\|_1 \left\|\mathbf{1}_{m,\ell} \check{f}_3\right\|_2 .$$

In order to obtain a similar estimate for $\Lambda_{j,m,n,\ell_0,\ell,1}(f_1, f_2, f_3)$, we change variables by $u = x - 2^m t$ and $v = 2^{(d-1)j}x - 2^m t^d$ to express $\Lambda_{j,m,n,\ell_0,\ell,1}(f_1, f_2, f_3)$ as

$$\iint \left(\mathbf{1}_{m,\ell+\ell_0} R_{\Phi_1} \check{f}_1\right)(u) R_{\Phi_1} \check{f}_2(v) \rho(t(u,v)) \left(\mathbf{1}^*_{(d-1)|j|+m,n} \mathbf{1}_{m,\ell} \check{f}_3\right)(x(u,v)) \frac{du dv}{J(u,v)},$$

where $J(u,v)$ is the Jocobian $\frac{\partial(u,v)}{\partial(x,t)}$. It is easy to see that the Jocobian $\frac{\partial(u,v)}{\partial(x,t)} \sim 2^m$. As we did for $\Lambda_{j,m,n,\ell_0,\ell,1}$, we dominate the previous integral by

$$C 2^{-m} \int \left\|\mathbf{1}_{m,\ell+\ell_0} R_{\Phi_1} \check{f}_1\right\|_2 \left|R_{\Phi_1} \check{f}_2(v)\right| \left(\int \left|\left(\mathbf{1}_{m,\ell} \check{f}_3\right)(x(u,v)) \rho(t(u,v))\right|^2 du\right)^{\frac{1}{2}} dv.$$

Notice that $|\partial x/\partial u| \sim 1$ whenever $t \in \mathrm{supp}\rho$. We then estimate

$$(9.8) \qquad |\Lambda_{j,m,n,\ell_0,\ell,1}(f_1, f_2, f_3)| \leq C 2^{-m} \left\|\mathbf{1}_{m,\ell+\ell_0} R_{\Phi_1} \check{f}_1\right\|_2 \left\|\check{f}_2\right\|_1 \left\|\mathbf{1}_{m,\ell} \check{f}_3\right\|_2 ,$$

(9.6) follows from (9.7) and (9.8). An interpolation of (9.3) and (9.6) then yields

$$(9.9) \qquad \left|\Lambda^{(1)}_{j,m,n}(f_1, f_2, f_3)\right| \leq C 2^{\frac{(d-1)j}{2} - \frac{m}{2}} 2^{\frac{(d-1)j+m}{2}} \prod_{i=1}^3 \|f_i\|_{L^2(\mathbf{I}_i)} .$$

We now turn to prove that if f_2 is σ-uniform in $\mathcal{Q}_2$, then

$$(9.10) \qquad \left|\Lambda^{(2)}_{j,m,n}(f_1, f_2, f_3)\right| \leq C_N 2^{\frac{(d-1)j}{2} - \frac{m}{2}} \max\left\{2^{-100m}, \sigma\right\} \prod_{i=1}^3 \|f_i\|_{L^2(\mathbf{I}_i)} .$$

In dual frequency variables, $\Lambda^{(2)}_{j,m,n}(f_1, f_2, f_3)$ can be expressed as

$$\sum_{\ell_0=-10}^{10} \sum_{\ell} 2^{\frac{(d-1)j}{2}} \iint \widehat{F_{1,m,\ell_0,\ell}}(\xi) \widehat{f_2(\eta) \widehat{\Phi_1}(\eta)} e^{2\pi i 2^{(d-1)j}\alpha_{m,\ell}\eta} \mathfrak{m}(\xi,\eta) \widehat{F_{3,m,n,\ell}}(\xi) d\xi d\eta ,$$

where

$$(9.11) \qquad \mathfrak{m}(\xi,\eta) = \int \rho(t) e^{-2\pi i(2^m \xi t + 2^m \eta t^d)} dt$$

$$F_{1,m,\ell_0,\ell} = \mathbf{1}_{m,\ell+\ell_0} R_{\Phi_1} \check{f}_1 \quad \text{and} \quad F_{3,m,n,\ell} = \mathbf{1}^*_{(d-1)|j|+m,n} \mathbf{1}_{m,\ell} \check{f}_3 .$$

If ξ is not in a small neighborhood of $\widehat{\Phi}_1$, then there is no critical point of the phse function $\phi_{\xi,\eta}(t) = \xi t + \eta t^d$ occurring in a small neighborhood of $\operatorname{supp}\rho$. Integration by parts gives a rapid decay $O(2^{-Nm})$ for $\mathfrak{m}$. Thus in this case, we dominate $\left|\Lambda^{(2)}_{j,m,n}(f_1,f_2,f_3)\right|$ by

$$(9.12) \qquad C_N 2^{-Nm} \prod_{i=1}^3 \|f_i\|_{L^2(\mathbf{I}_i)},$$

for any positive integer N. We now only need to consider the worst case when there is a critical point of the phse function $\phi_{\xi,\eta}(t) = \xi t + \eta t^d$ in a small neighborhood of $\operatorname{supp}\rho$. In this case, ξ must be in a small neighborhood of $\widehat{\Phi}_1$ and the stationary phase method gives

$$(9.13) \qquad \mathfrak{m}(\xi,\eta) \sim 2^{-m/2} e^{2\pi i c_d 2^m \xi^{d/(d-1)} \eta^{-\frac{1}{d-1}}},$$

where c_d is a constant depending on d only. Thus the principle term of $\Lambda^{(2)}_{j,m,n}(f_1,f_2,f_3)$ is

$$\sum_{\ell_0=-10}^{10}\sum_\ell 2^{\frac{(d-1)j}{2}-\frac{m}{2}} \iint \widehat{F_{1,m,\ell_0,\ell}}(\xi)\widehat{\Phi}_2(\xi) f_2(\eta)\widehat{\Phi}_1(\eta) e^{i\phi_{d,m,\xi}(\eta)} \widehat{F_{3,m,n,\ell}}(\xi) d\xi d\eta,$$

where $\widehat{\Phi}_2$ is a Schwartz function supported on a small neighborhood of $\widehat{\Phi}_1$, and

$$\phi_{d,m,\xi}(\eta) = 2\pi c_d 2^m \xi^{d/(d-1)} \eta^{-\frac{1}{d-1}} + 2\pi 2^{(d-1)j}\alpha_{m,\ell}\eta.$$

The key point is that the integral in the previous expression can be viewed as an inner product of $F_{3,m,n,\ell}$ and $\mathcal{M}F_{1,m,\ell_0,\ell}$, where $\mathcal{M}$ is a multiplier operator defined by

$$\widehat{\mathcal{M}f}(\xi) = \mathfrak{m}_{d,j,m}(\xi)\widehat{f}(\xi).$$

Here the multiplier $\mathfrak{m}_{d,j,m}$ is given by

$$(9.14) \qquad \mathfrak{m}_{d,j,m}(\xi) = \int f_2(\eta)\widehat{\Phi}_1(\eta) e^{i\phi_{d,m,\xi}(\eta)} d\eta.$$

Observe that $\phi_{d,m,\xi}(\eta)+b\eta$ is in $\mathcal{Q}_2$ for any $b\in\mathbb{R}$ and $\xi\in\operatorname{supp}\widehat{\Phi}_2$. Thus σ-uniformity in $\mathcal{Q}_2$ of f_2 yields

$$(9.15) \qquad \|\mathfrak{m}_{d,j,m}\|_\infty \le C\sigma\,\|f_2\|_{L^2(\mathbf{I}_2)}.$$

And henceforth we dominate $\Lambda^{(2)}_{j,m,n}(f_1,f_2,f_3)$ by

$$\sum_{\ell_0=-10}^{10}\sum_\ell 2^{\frac{(d-1)j}{2}-\frac{m}{2}}\sigma\,\|f_2\|_{L^2(\mathbf{I}_2)}\|F_{1,m,\ell_0,\ell}\|_2\|F_{3,m,n,\ell}\|_2,$$

which clearly is bounded by

$$(9.16) \qquad 2^{\frac{(d-1)j}{2}-\frac{m}{2}}\sigma\prod_{i=1}^3 \|f_i\|_{L^2(\mathbf{I}_i)}.$$

Now (9.10) follows from (9.12) and (9.16). Combining (9.9) and (9.10), we finish the proof. $\qquad\square$

Corollary 9.1. *Let $j \leq 0$ and $\Lambda_{j,m,n}(f_1, f_2, f_3)$ be defined as in (5.19). Then there exists a constant C independent of j, m, n such that*

$$(9.17) \quad |\Lambda_{j,m,n}(f_1, f_2, f_3)| \leq C \max\left\{2^{-100m}, 2^{\frac{(d-1)j+m}{2}}, \sigma\right\} \|f_1\|_{L^2(\mathbf{I}_1)} \|f_2\|_{L^2(\mathbf{I}_1)} \|\widehat{f_3}\|_\infty,$$

holds for all $f_2 \in L^2(\mathbf{I}_2)$ which are σ-uniform in $\mathcal{Q}_2$, $f_1 \in L^2(\mathbf{I}_1)$ and $\widehat{f_3} \in L^\infty$.

Proof. Since there is a smooth restriction factor $\mathbf{1}^*_{(d-1)|j|+m,n}$ in the definition of $\mathcal{B}_{j,m,m}$, the right hand side of (9.2) can be sharpen to

$$(9.18) \quad C 2^{\frac{(d-1)j}{2} - \frac{m}{2}} \max\left\{2^{-100m}, 2^{\frac{(d-1)j+m}{2}}, \sigma\right\} \|f_1\|_{L^2(\mathbf{I}_1)} \|f_2\|_{L^2(\mathbf{I}_2)} \left\|\mathbf{1}^{**}_{(d-1)|j|+m,n} \check{f_3}\right\|_2,$$

which is clearly bounded by

$$C \max\left\{2^{-100m}, 2^{\frac{(d-1)j+m}{2}}, \sigma\right\} \|f_1\|_{L^2(\mathbf{I}_1)} \|f_2\|_{L^2(\mathbf{I}_2)} \|\widehat{f_3}\|_\infty.$$

$\square$

Proposition 9.2. *Let $j \leq 0$ and $\Lambda_{j,m,n}(f_1, f_2, f_3)$ be defined as in (5.19). Then there exists a constant C independent of j, m, n such that*

$$(9.19) \quad \left|\Lambda_{j,m,n}(f_1, e^{iq_2}, f_3)\right| \leq C 2^{-m/4} \|f_1\|_{L^2(\mathbf{I}_1)} \|\widehat{f_3}\|_\infty,$$

holds for all $q_2 \in \mathcal{Q}_2$, $f_1 \in L^2(\mathbf{I}_1)$ and $\widehat{f_3} \in L^\infty$.

We will prove Proposition 9.2 in Section 11. We now are ready to provide a proof of Theorem 5.4.

9.1. **Proof of Theorem 5.4.**

Corollary 8.1, Proposition 8.2 and Theorem 7.1 yield that $|\Lambda_{j,m,n}(f_1, f_2, f_3)|$ is dominated by

$$(9.20) \quad C\left(\max\left\{2^{-100m}, 2^{\frac{-(d-1)j+m}{2}}, \sigma\right\} + \frac{2^{-\mathfrak{D}(d-1)m/2}}{\sigma}\right) \|f_1\|_{L^2(\mathbf{I}_1)} \|f_2\|_{L^2(\mathbf{I}_1)} \|\widehat{f_3}\|_\infty,$$

holds for all $f_1 \in L^2(\mathbf{I}_1)$, $f_2 \in L^2(\mathbf{I}_2)$ and $\widehat{f_3} \in L^\infty$. Take σ to be $2^{-\mathfrak{D}(d-1)m/4}$, then we have
$$(9.21)$$
$$|\Lambda_{j,m,n}(f_1, f_2, f_3)| \leq C \max\left\{2^{\frac{-(d-1)j+m}{2}}, 2^{-\mathfrak{D}(d-1)m/4}\right\} \|f_1\|_{L^2(\mathbf{I}_1)} \|f_2\|_{L^2(\mathbf{I}_1)} \|\widehat{f_3}\|_\infty.$$

This give a proof for the case $j > 0$. For the case $j \leq 0$, applying Corollary 9.1, Proposition 9.2 and Theorem 7.1, we estimate $|\Lambda_{j,m,n}(f_1, f_2, f_3)|$ by

$$(9.22) \quad C\left(\max\left\{2^{-100m}, 2^{\frac{(d-1)j+m}{2}}, \sigma\right\} + \frac{2^{-m/4}}{\sigma}\right) \|f_1\|_{L^2(\mathbf{I}_1)} \|f_2\|_{L^2(\mathbf{I}_1)} \|\widehat{f_3}\|_\infty,$$

holds for all $f_1 \in L^2(\mathbf{I}_1)$, $f_2 \in L^2(\mathbf{I}_2)$ and $\widehat{f_3} \in L^\infty$. Now choose σ to be $2^{-m/8}$. Then we have

$$(9.23) \quad |\Lambda_{j,m,n}(f_1, f_2, f_3)| \leq C \max\left\{2^{\frac{(d-1)j+m}{2}}, 2^{-m/8}\right\} \|f_1\|_{L^2(\mathbf{I}_1)} \|f_2\|_{L^2(\mathbf{I}_1)} \|\widehat{f_3}\|_\infty,$$

which completes the proof of the case $j \leq 0$. Therefore combining (9.21) and (9.23), we proved Theorem 5.4.

10. Proof of Proposition 8.2

Lemma 10.1. *Let $\ell \geq 1$. Let $\mathbf{I}_1$ and $\mathbf{I}_2$ be fixed bounded intervals. And let φ be a function from $\mathbf{I}_1 \times \mathbf{I}_2$ to $\mathbb{R}$ satisfying*

$$(10.1) \qquad \left| \partial_x^\ell \partial_y \varphi(x,y) \right| \geq 1 \,,$$

for all $(x,y) \in \mathbf{I}_1 \times \mathbf{I}_2$. Assume an additional condition holds in the case $\ell = 1$,

$$(10.2) \qquad \left| \partial_x^2 \partial_y \varphi(x,y) \right| \neq 0 \,,$$

for all $(x,y) \in \mathbf{I}_1 \times \mathbf{I}_2$. Then there exists a constant depending on the length of $\mathbf{I}_1$ and $\mathbf{I}_2$ but independent of φ, λ and the locations of $\mathbf{I}_1$ and $\mathbf{I}_2$ such that

$$(10.3) \qquad \left| \iint_{\mathbf{I}_1 \times \mathbf{I}_2} e^{i\lambda\varphi(x,y)} f(x)g(x)dxdy \right| \leq C(1+|\lambda|)^{-\mathfrak{D}(\ell)} \|f\|_2 \|g\|_2 \,,$$

for all $f, g \in L^2$, where

$$(10.4) \qquad \mathfrak{D}(\ell) = \begin{cases} 1/(2\ell), & \text{if } \ell \geq 2 \,; \\ 1/(2+\varepsilon), & \text{if } \ell = 1 \,. \end{cases}$$

for any $\varepsilon > 0$.

This lemma is related to a 2-dimensional van der Corput lemma proved in [1]. The case $\ell \geq 2$ was proved in [1]. And a proof of the case $\ell = 1$ can be found in [23]. The estimates on $\mathfrak{D}(\ell)$ in (10.4) are not sharp. With some additional convexity conditions on the phase function φ, $\mathfrak{D}(\ell)$ might be improved to be $1/(\ell+1)$ (see [1] for some of such improvements). But in this article we do not need to pursue the sharp estimates.

Lemma 10.2. *Let $c, \tau \in \mathbb{R}$ and φ be a function defined by*

$$(10.5) \qquad \varphi_c(x,y) = \left(x - y^{1/d} + c \right)^d$$

Define $\mathbf{Q}_{c,j,\tau}(x,y)$ by

$$(10.6) \qquad \mathbf{Q}_{c,j,\tau}(x,y) = \varphi_c(x,y) - \varphi_c(x + 2^{-(d-1)j}\tau, y + \tau) \,.$$

Then there exists a constant C_d depending only on d such that

$$(10.7) \qquad \left| \partial_x^{d-1} \partial_y \mathbf{Q}_{c,j,\tau}(x,y) \right| \geq C_d |\tau|$$

holds for all $y, y + \tau \in [2^{-100}, 2^{100}]$. Moreover, if $d = 2$, we have

$$(10.8) \qquad \left| \partial_x \partial_y^2 \mathbf{Q}_{c,j,\tau}(x,y) \right| \geq C_d |\tau|$$

holds for all $y, y + \tau \in [2^{-100}, 2^{100}]$.

Proof. A direct computation yields

$$(10.9) \qquad \partial_x^{d-1} \partial_y \mathbf{Q}_{c,j,\tau}(x,y) = C_d \left((y+\tau)^{\frac{1}{d}-1} - y^{\frac{1}{d}-1} \right) \,.$$

Hence the desired estimate (10.7) follows immediately from the mean value theorem. (10.8) can be obtained similarly. $\qquad \square$

Lemma 10.3. *Let* $\mathbf{I}$ *be a fixed interval of length* 1. *And let* θ *be a bump function supported on* $[1/100, 2]$ *(or* $[-2, -1/100]$*). Suppose that* $\phi_{d,j,m}$ *is a phase function defined by*

$$(10.10) \qquad \phi_{d,j,m}(x,y) = C_{d,j,m} 2^m \left(x - y^{\frac{1}{d}} + c_{j,m} \right)^d ,$$

where $C_{d,j,m}, c_{j,m}$ *are constants independent of* x, y *such that* $2^{-200} \leq |C_{d,j,m}| \leq 2^{200}$. *Let* $\Lambda_{d,j,m,\mathbf{I}}$ *be a bilinear form defined by*

$$(10.11) \qquad \Lambda_{d,j,m,\mathbf{I}}(f,g) = \iint e^{i\phi_{d,j,m}(x,t)} f\left(x - 2^{-(d-1)j} t \right) g(x) \mathbf{1}_{\mathbf{I}}(x) \theta(t) dx dt .$$

Then we have

$$(10.12) \qquad |\Lambda_{d,j,m,\mathbf{I}}(f,g)| \leq C_d 2^{-\frac{\mathfrak{D}(d-1)m}{2}} \|f\|_2 \|g\|_\infty ,$$

holds for all $f \in L^2$ *and* $g \in L^\infty$, *where* C_d *is a constant depending only on* d.

Proof. The bilinear form $\Lambda_{d,j,m,\mathbf{I}}(f,g)$ equals to $\langle \mathbf{T}_{d,j,m,\mathbf{I}}(g), f \rangle$, where $\mathbf{T}_{d,j,m,\mathbf{I}}$ is defined by

$$(10.13) \qquad \mathbf{T}_{d,j,m,\mathbf{I}} g(x) = \int e^{i\phi_{d,j,m}(x+2^{-(d-1)j}t, t)} (g\mathbf{1}_{\mathbf{I}}) \left(x + 2^{-(d-1)j} t \right) \theta(t) dt .$$

By a change of variables, $\|\mathbf{T}_{d,j,m,\mathbf{I}} g\|_2^2$ can be expressed as

$$\int \left(\iint e^{i\Phi_{d,j,m,\tau}(x,t)} G_\tau \left(x + 2^{-(d-1)j} t \right) \Theta_\tau(t) dx dt \right) d\tau ,$$

where

$$\Phi_{d,j,m,\tau}(x,t) = \phi_{d,j,m}(x + 2^{-(d-1)j} t, t) - \phi_{d,j,m}(x + 2^{-(d-1)j} t + 2^{-(d-1)j} \tau, t + \tau)$$

$$G_\tau(x) = (\mathbf{1}_{\mathbf{I}} g)(x) \overline{(\mathbf{1}_{\mathbf{I}} g) \left(x + 2^{-(d-1)j} \tau \right)} ,$$

$$\Theta_\tau(t) = \theta(t) \theta(t + \tau) .$$

Changing coordinates $(x,t) \mapsto (u,v)$ by $u = x + 2^{-(d-1)j} t$ and $v = t$, we write the inner double-integral in the previous integral as

$$\iint e^{iC_{d,j,m} 2^m \mathbf{Q}_{c_{j,m},j,\tau}(u,v)} G_\tau(u) \Theta_\tau(v) du dv ,$$

where $\mathbf{Q}_{c_{j,m},j,\tau}$ is defined as in (10.6). From (10.7), (10.8) and Lemma 10.1, we then estimate $\|\mathbf{T}_{d,j,m,\mathbf{I}} g\|_2^2$ by

$$C_d \int_{-10}^{10} \min\left\{ 1, 2^{-\mathfrak{D}(d-1)m} \tau^{-\mathfrak{D}(d-1)} \right\} \|G_\tau\|_2 \|\Theta_\tau\|_2 \, d\tau ,$$

which clearly is bounded by

$$C_d 2^{-\mathfrak{D}(d-1)m} \|g\|_\infty^2 ,$$

Hence (10.12) follows and therefore we complete the proof. $\qquad\qquad\qquad\square$

We now turn to the proof of Proposition 8.2. For simplicity, we assume ρ is supported on $[1/8, 2]$. For any function $q_1 = (a\xi^{\frac{d}{d-1}} + b\xi \in \mathcal{Q}_1$, we have

$$(10.14) \qquad R_{\Phi_1}(e^{\check{i}q_1})(x) = \int \widehat{\Phi_1}(\xi) e^{ia\xi^{\frac{d}{d-1}}} e^{i(x+b)\xi} d\xi \,,$$

where $|a| \sim 2^m$. The stationary phase method yields that the principle part of (10.14) is

$$(10.15) \qquad \mathcal{P}(q_1)(x) = C_d|a|^{-1/2} e^{ic_1 a^{-(d-1)}(x+b)^d} \widehat{\Phi_1}\left(c_2 a^{-(d-1)}(x+b)^{d-1}\right) \,,$$

where C_d, c_1, c_2 are constants depending only on d. Thus to obtain Proposition 8.2, it suffices to prove that there exists a constant C such that

$$(10.16) \qquad \left|\tilde{\Lambda}_{j,m,n}(e^{iq_1}, f_2, f_3)\right| \le C2^{-\frac{\mathfrak{D}(d-1)m}{2}} \|\check{f}_2\|_2 \|\check{f}_3\|_\infty \,,$$

holds for all $q_1 \in \mathcal{Q}_1$, $\check{f}_2 \in L^2$, and $\check{f}_3 \in L^\infty$, where $\tilde{\Lambda}_{j,m,n}(e^{iq_1}, f_2, f_3)$ is defined to be

$$2^{-\frac{(d-1)j}{2}} \iint \mathcal{P}(q_1)\left(2^{-(d-1)j}x - 2^m t\right) \check{f}_2\left(x - 2^m t^d\right) \left(1^*_{(d-1)j+m,n}\check{f}_3\right)(x)\rho(t)dtdx \,.$$

Observe that $\widehat{\Phi_1}$ is supported essentially in a bounded interval away from 0. Thus we can restrict the variable x in a bounded interval $\mathbf{I}_{d,j,m}$ whose length is comparable to $2^{(d-1)j+m}$ and reduce the problem to estimate

$$(10.17) \qquad \left|\Lambda_{j,m,n,\mathbf{I}_{d,j,m}}(f_2, f_3)\right| \le C2^{-\frac{\mathfrak{D}(d-1)m}{2}} \|\check{f}_2\|_2 \|\check{f}_3\|_\infty \,,$$

holds for an absolute constant C and all $\check{f}_2 \in L^2$, $\check{f}_3 \in L^\infty$, where $\Lambda_{j,m,n,\mathbf{I}_{d,j,m}}(f_2, f_3)$ is equal to

$$(10.18) \quad 2^{-\frac{(d-1)j}{2}-\frac{m}{2}} \iint \mathcal{P}_{d,j,m}\left(2^{-(d-1)j}x - 2^m t\right) \check{f}_2\left(x - 2^m t^d\right) \left(1_{\mathbf{I}_{d,j,m}}\check{f}_3\right)(x)\rho(t)dtdx \,.$$

Here

$$(10.19) \qquad \mathcal{P}_{d,j,m}(x) = e^{ic_1 a^{-(d-1)}(x+b)^d} \widehat{\Phi_1}\left(c_2 a^{-(d-1)}(x+b)^{d-1}\right) \,.$$

Let $\mathbf{I}$ be an interval of length 1. A rescaling argument then reduces (10.17) to an estimate of a bilinear form $\Lambda_{j,m,n,\mathbf{I}}$ associated to $\mathbf{I}$, that is,

$$(10.20) \qquad |\Lambda_{j,m,n,\mathbf{I}}(f, g)| \le C2^{-\frac{\mathfrak{D}(d-1)m}{2}} \|f\|_2 \|g\|_\infty \,,$$

where $\Lambda_{j,m,n,\mathbf{I}}(f, g)$ is defined by

$$\iint \mathcal{P}_{d,j,m}\left(2^m x - 2^m t\right) f\left(x - 2^{-(d-1)j}t^d\right) g(x)\mathbf{1}_{\mathbf{I}}(x)\rho(t)dtdx \,.$$

Notice that

$$(10.21) \qquad \mathcal{P}_{d,j,m}\left(2^m x - 2^m t\right) = e^{iC_{d,j,m}2^m(x-t+c_{j,m})^d} \widehat{\Phi_1}\left(C_d C_{d,m}(x - t + c_m)^{d-1}\right) \,,$$

where $C_{d,j,m}, C_{d,m}, c_{j,m}, c_m, C_d$ are constants such that $|C_{d,j,m}|, |C_{d,m}| \in [2^{-100}, 2^{100}]$. $\widehat{\Phi_1}\left(C_d C_{d,m}(x - t + c_m)^{d-1}\right)$ can be dropped by utilizing Fourier series since $\widehat{\Phi_1}$ is a Schwartz function, because $x \in \mathbf{I}, t \in \text{supp}\,\rho$ are restricted in bounded intervals respectively. Then (10.20) can be reduced to Lemma 10.3 by a change of variable $t^d \mapsto t$.

Therefore we prove Proposition 8.2.

11. Proof of Proposition 9.2

Lemma 11.1. *Let $j \leq 0$ and $\tau \in [-100, 100]$. And let $\phi_{d,j,m,\tau}$ be defined by*

$$(11.1) \qquad \phi_{d,j,m,\tau}(u, v) = \left(u - v^d\right)^{1/d} - \left(u + 2^{(d-1)j}\tau - (v + \tau)^d\right)^{1/d}.$$

Suppose that $|j| \geq m/(d-1)$ and $|u| \geq 2^{-m}$. Then

$$(11.2) \qquad |\partial_v \phi_{d,j,m,\tau}(u, v)| \geq C|\tau u|$$

holds whenever $v, v + \tau \in \mathbf{I}_1$ and $u - v^d, u - (v + \tau)^d \in \mathbf{I}_2$, where $\mathbf{I}_i$ is the interval $[1/100, 100]$ or $[-100, -1/100]$.

Proof. Clearly

$$\partial_v \phi_{d,j,m,\tau}(u, v) = -\left(u - v^d\right)^{\frac{1}{d}-1} v^{d-1} + \left(u + 2^{(d-1)j}\tau - (v + \tau)^d\right)^{\frac{1}{d}-1}(v + \tau)^{d-1},$$

which can be written as a sum

$$\Phi_{d,j,m,\tau,1}(u, v) + \Phi_{d,j,m,\tau,2}(u, v),$$

where $\Phi_{d,j,m,\tau,1}(u, v)$ is

$$-\left(u - v^d\right)^{\frac{1}{d}-1} v^{d-1} + \left(u - (v + \tau)^d\right)^{\frac{1}{d}-1}(v + \tau)^{d-1}$$

and $\Phi_{d,j,m,\tau,2}(u, v)$ is equal to

$$-\left(u + 2^{(d-1)j}\tau - (v + \tau)^d\right)^{\frac{1}{d}-1}(v + \tau)^{d-1} + \left(u - (v + \tau)^d\right)^{\frac{1}{d}-1}(v + \tau)^{d-1}.$$

The mean value theorem yields that

$$(11.3) \qquad |\Phi_{d,j,m,\tau,2}(u, v)| \leq C2^{(d-1)j}|\tau|,$$

and

$$\Phi_{d,j,m,\tau,1}(u, v) = G'_{d,j,m,\tau}(\eta)\tau,$$

where

$$(11.4) \qquad G_{d,j,m,\tau}(v) = -\left(u - v^d\right)^{\frac{1}{d}-1} v^{d-1}$$

and η is a point between v and $v + \tau$. A simple computation gives

$$(11.5) \qquad G'_{d,j,m,\tau}(v) = -(d-1)\left(u - v^d\right)^{\frac{1}{d}-2} v^{d-2}u.$$

Now (11.2) follows from (11.3), (11.5), $|j| \geq m/(d-1)$ and $|u| \geq 2^{-m}$. Therefore we finish the proof. $\square$

Lemma 11.2. *Let θ_1, θ_2 be bump functions supported on $\mathbf{I}_1$ and $\mathbf{I}_2$ respectively, where $\mathbf{I}_i$ is $[1/50, 2]$ or $[-2, -1/50]$. Suppose that $j \leq 0$, $|j| \geq m/(d-1)$ and $\phi_{d,j,m}$ is a phase function defined by*

$$(11.6) \qquad \phi_{d,j,m}(x, t) = C_{d,j,m} 2^m \left(x - t^d\right)^{1/d},$$

where $C_{d,j,m}$ are constants independent of x,t such that $2^{-200} \le |C_{d,j,m}| \le 2^{200}$. Let $\Lambda_{d,j,m}$ be a bilinear form defined by

$$(11.7) \qquad \Lambda_{d,j,m}(f,g) = \iint e^{i\phi_{d,j,m}(x,t)} f\left(x - 2^{(d-1)j}t\right) g(x)\theta_1(x - t^d)\theta_2(t)dxdt.$$

Then we have

$$(11.8) \qquad\qquad |\Lambda_{d,j,m}(f,g)| \le C_d 2^{-m/4}\|f\|_2\|g\|_\infty,$$

holds for all $f \in L^2$ and $g \in L^\infty$, where C_d is a constant depending only on d.

Proof. The bilinear form $\Lambda_{d,j,m}(f,g)$ equals to $\langle \mathbf{T}_{d,j,m}(g), f\rangle$, where $\mathbf{T}_{d,j,m}$ is defined by

$$(11.9) \quad \mathbf{T}_{d,j,m}g(x) = \int e^{i\phi_{d,j,m}(x+2^{(d-1)j}t,t)}g\left(x + 2^{(d-1)j}t\right)\theta_1\left(x + 2^{(d-1)j}t - t^d\right)\theta_2(t)dt.$$

By a change of variables, we express $\|\mathbf{T}_{d,j,m}g\|_2^2$ as

$$\int\left(\iint e^{i\Phi_{d,j,m,\tau}(x+2^{(d-1)j}t,t)}G_\tau\left(x + 2^{(d-1)j}t\right)\Theta_{1,\tau}\left(x + 2^{(d-1)j}t, t\right)\Theta_{2,\tau}(t)dxdt\right)d\tau,$$

where

$$\Phi_{d,j,m,\tau}(x,t) = \phi_{d,j,m}(x,t) - \phi_{d,j,m}(x + 2^{(d-1)j}\tau, t + \tau)$$

$$G_\tau(x) = g(x)\overline{g\left(x + 2^{(d-1)j}\tau\right))},$$

$$\Theta_{1,\tau}(x,t) = \theta_1\left(x - t^d\right)\overline{\theta_1\left(x + 2^{(d-1)j}\tau - (t+\tau)^d\right)},$$

$$\Theta_{2,\tau}(t) = \theta_2(t)\theta_2(t + \tau).$$

Changing coordinates $(x,t) \mapsto (u,v)$ by $u = x + 2^{(d-1)j}t$ and $v = t$, we write the inner double-integral in the previous integral as

$$(11.10) \qquad\qquad \iint e^{i\Phi_{d,j,m,\tau}(u,v)}G_\tau(u)\Theta_{1,\tau}(u,v)\Theta_{2,\tau}(v)dudv.$$

It is clear that $\partial_v^2 \Phi_{d,j,m,\tau}$ has at most finite zeros for fixed τ, u. Thus, by Lemma 11.1 and van der Corput lemma, we obtain

$$(11.11) \qquad \left|\int e^{i\Phi_{d,j,m,\tau}(u,v)}\Theta_{1,\tau}(u,v)\Theta_{2,\tau}(v)dv\right| \le C\min\left\{1, 2^{-m}|\tau u|^{-1}\right\}$$

holds for $|u| \ge 2^{-m}$. Thus we estimate $\|\mathbf{T}_{d,j,m}g\|_2^2$ by

$$C2^{-m}\|g\|_\infty^2 + C_d \int_{-10}^{10}\int_{-10}^{10} \min\left\{1, 2^{-m/2}|\tau u|^{-1/2}\right\}\|G_\tau\|_\infty \, dud\tau,$$

which is dominated by

$$C_d 2^{-m/2}\|g\|_\infty^2.$$

Henceforth, (11.8) follows and we complete the proof. $\qquad\qquad\qquad\square$

We now turn to the proof of Proposition 9.2. For simplicity, we assume ρ is supported on $[1/8, 2]$. For any function $q_2 = a\eta^{-\frac{1}{d-1}} + b\eta \in \mathcal{Q}_2$, we have

$$(11.12) \qquad R_{\Phi_1}(e^{\check{i}q_2})(x) = \int \widehat{\Phi_1}(\eta)e^{ia\eta^{-\frac{1}{d-1}}}e^{i(x+b)\eta}d\eta,$$

where $|a| \sim 2^m$. By the stationary phase method, we obtain that the principle part of (11.12) is

$$(11.13) \qquad \mathcal{P}(q_2)(x) = C_d|a|^{-1/2}e^{ic_1 a^{(d-1)/d}(x+b)^{1/d}}\widehat{\Phi_1}\left(c_2\left(\frac{x+b}{a}\right)^{-\frac{d-1}{d}}\right),$$

where C_d, c_1, c_2 are constants depending only on d. Thus to obtain Proposition 8.2, it suffices to prove that there exists a constant C such that

$$(11.14) \qquad \left|\tilde{\Lambda}_{j,m,n}(f_1, e^{iq_2}, f_3)\right| \leq C2^{-m/4}\|\check{f}_1\|_2\|\check{f}_3\|_\infty,$$

holds for all $q_2 \in \mathcal{Q}_2$, $\check{f}_1 \in L^2$, and $\check{f}_3 \in L^\infty$, where $\tilde{\Lambda}_{j,m,n}(f_1, e^{iq_2}, f_3)$ equals to

$$2^{\frac{(d-1)j}{2}}\iint \check{f}_1\left(x - 2^m t\right)\mathcal{P}(q_2)\left(2^{(d-1)j}x - 2^m t^d\right)\check{f}_3(x)\rho(t)dtdx.$$

By a rescaling argument, it suffices to show that

$$(11.15) \qquad \tilde{\Lambda}_{d,j,m}(f, g) \leq C2^{-m/4}\|f\|_2\|g\|_\infty$$

holds for all $f \in L^2$ and $g \in L^\infty$, where $\tilde{\lambda}_{d,j,m}$ is defined by
(11.16)
$$\tilde{\Lambda}_{d,j,m}(f, g) = \iint f\left(x - 2^{(d-1)j}t\right)g(x)e^{iC_{d,j,m}2^m(x-t^d+b/2^m)^{1/d}}\theta_1(x - t^d + b/2^m)\rho(t)dtdx.$$

Here the constant $C_{d,j,m}$ satisfies $2^{-100} \leq |C_{d,j,m}| \leq 2^{100}$ and θ_1 is a bump function supported on $[1/100, 2]$ or $[-2, -1/100]$. Clearly (11.15) is a consequence of Lemma 11.2. Therefore the proof of Proposition 9.2 is completed.

REFERENCES

[1] A. Carbery, M. Christ and J. Wright, *Multidimensional van der Corput and sublevel set estimates*, J. Amer. Math. Soc. **12** (1999), 981-1015.

[2] M. Christ, *Hilbert transforms along curves, I: Nilpotent groups*, Ann. Math. **122** (1985), 575-596.

[3] M. Christ, *Hilbert transforms along curves, II: A flat case*, Duke Math. J. **52** (1985), 887-894.

[4] M. Christ, X. Li, T. Tao and C. M. Thiele, *On multilinear oscillatory integrals, nonsingular and singular*, Duke Math. J. **130** (2005), 321-351.

[5] M. Christ, A. Nagel, E. Stein and S. Wainger, *Singular and maximal Radon transforms: analysis and geometry*, Ann of Math. 150 (1999), 489-577.

[6] J. Duoandikoetxea and J. L. Rubio de Francia, Maximal and singular integral operators via Fourier transform estimates, Invent. Math. **84** (1986), 541-561.

[7] E.B. Fabes and N.M. Riviere, *Singular integral with mixed homogeneity*, Studia Math. **27** (1966), 19-38.

[8] D. Fan and X. Li, *A bilinear oscillatory integral along parabolas*, submitted.

[9] H. Furstenberg, *Nonconventional ergodic averages*, Proceedings of Symposia in Pure Math., **50** (1990), 43–56

[10] W. T. Gowers, *A new proof of Szemerédi's theorem for arithmetic progressions of length four*, Geom. Funct. Anal. **8** (1998), 529-551.

[11] L. Grafakos and X. Li, *The disc as a bilinear multiplier*, Amer. J. of Math. **128** (2006), 91-119.

[12] L. Grafakos and X. Li, *Uniform bounds for the bilinear Hilbert transforms, I*, Ann. of Math. **159** (2004), 889-993.

[13] L. Hörmander, *Oscillatory integrals and multipliers on FL^p* , Ark. Mat., **11** (1973), 1–11.

[14] B. Host and B. Kra, *Nonconventional ergodic averages and nilmanifolds*, Ann. of Math. (2) 161 (2005), no. 1, 397–488.

[15] X. Li, *Uniform Estimates for Some Paraproducts*, N. Y. Math. J., to appear.

[16] M. T. Lacey, The bilinear maximal function maps into L^p, $p > 2/3$, Ann. Math. **151** (2000), no.1, 35-57

[17] M. T. Lacey and C. M. Thiele, *L^p estimates on the bilinear Hilbert transform for $2 < p < \infty$*, Ann. Math. **146** (1997), 693-724.

[18] M. T. Lacey and C. M. Thiele, *On Calderón's conjecture*, Ann. Math. **149** (1999), 475-496.

[19] X. Li, *Uniform bounds for the bilinear Hilbert transforms, II*, Rev. Mat. Iberoamericana **22** (2006), no.3, 1067-1124.

[20] C. Muscalu, T. Tao, and C. M. Thiele, *Uniform estimates on paraproducts*, Journal d'Analyse de Jerusalem **87** (2002), 369-384.

[21] C. Muscalu, T. Tao and C. M. Thiele, *Multilinear operators given by singular multipliers*, J. Amer. Math. Soc. **15** (2002), 469-496.

[22] A. Nagel, I. Vance, S. Wainger, and D. Weinberg, *Hilbert transforms for convex curves*, Duke. Math. J. **50** (1983), 735-744.

[23] D. H. Phong and E. M. Stein, *On a stopping process for oscillatory integrals* , J. Geom. Anal. **4** (1994), 104–120.

[24] C. D. Sogge, *Fourier integrals in classical analysis*, Cambridge University Press, 1993.

[25] E. Stein, *Harmonic analysis, real-variable methods, orthogonality, and oscillatory integrals*, Princeton, (1993).

[26] E. Stein, *Oscillatory integrals related to Radon-like transforms*, J. of Fourier Analysis and Appl., Kahane special issue, (1995), 535–551.

[27] E. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. **84** (1978), 1239-1295.

[28] C. M. Thiele, *A uniform estimate*, Ann. of Math. **157** (2002), 1-45.

Xiaochun Li, Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL, 61801, USA

E-mail address: xcli@math.uiuc.edu